

M335 GT 3,4

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics



STUDIES IN GEOMETRIC TOPOLOGY

UNITS 3,4

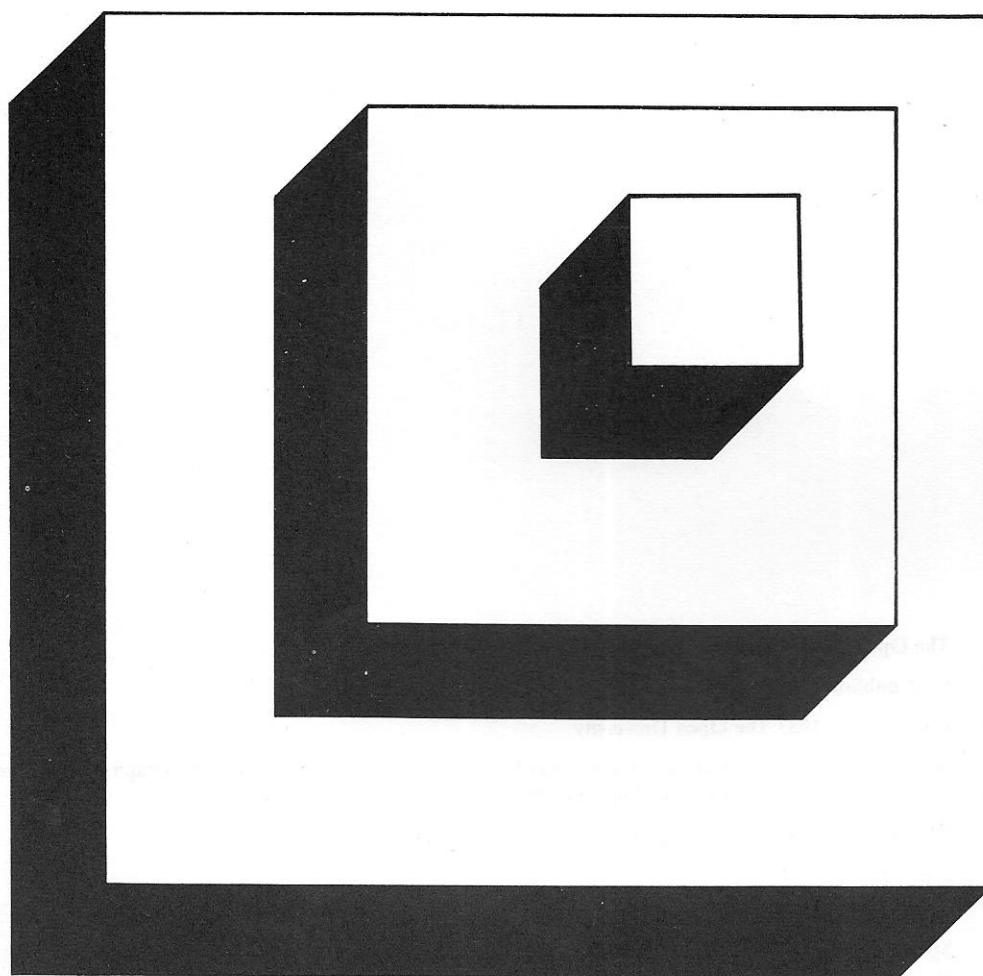
PURE MATHEMATICS





GEOMETRIC TOPOLOGY

UNITS 3, 4





THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M333: Statistics in Pure Mathematics
Prepared by the Course Team

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UN COMBINATORIAL SURFA

3.1 Combinatorial Surfaces

We are now going to spend some time studying the classification theorem for surfaces. As we hinted when giving the definition in *Unit 1*, the definition of a surface as a “locally disc-like” space leads to some unexpected complications. Since these complications do not add much to our understanding of what a surface is, we shall assume that a surface has already been subdivided into faces, edges and vertices. We have seen how to subdivide familiar surfaces like the sphere, torus and projective plane in this way, so this is a reasonable starting place. In the rest of this unit and in the next we shall prove that *subdivided* surfaces are classified by the Euler characteristic, number of boundary curves, and whether or not they are orientable, as stated in the classification theorem of *Unit 1*. The assumption that surfaces *can* be subdivided is certainly plausible: we shall return to this question at the end of *Unit 4*.

Recall that when a surface is subdivided into faces, edges and vertices, we require each face to be a topological disc with the boundary made up of edges and vertices. We now work in the opposite direction and give conditions for building up surfaces from faces by glueing them together along the edges. In the next reading passage, **B** explains exactly what we have to allow as faces (referred to as polygons in the text) to cater for the possibility that points on the boundary are identified. He then defines a surface as a set that can be divided into, or built up from, these faces.

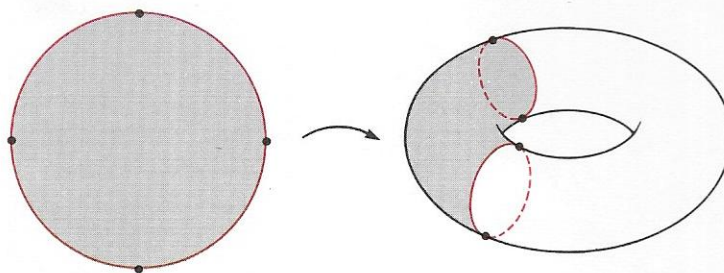
READ B Section 2.1 on page 53 to the end of page 55.

NOTES (i) *Page 53, line 9*

These “polygons” are, of course, homeomorphic to the straight-sided regular polygons for $n \geq 3$.

(ii) *Page 53, line -2*

B calls an edge of a topological polygon an “arc”, but it need not be a *circular* arc: it is just a homeomorphic image of a circular arc. The diagram below shows a topological four-sided polygon lying on a torus.



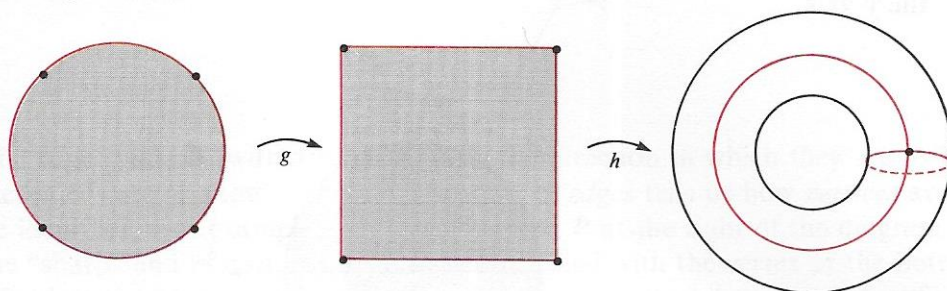
(iii) *Page 55, line 1*

It is not difficult to justify **B**'s assertion that a subdivision of a torus into topological polygons must have at least four polygons. In the usual representation of a torus as a rectangle with identified edges, the four vertices of the rectangle all represent the same point. Therefore, since no point may appear more than once as the vertex of a single topological polygon, these vertices must be placed in different polygons in the subdivision.

(iv) *Page 55, line 13* (Definition of singular topological polygon)

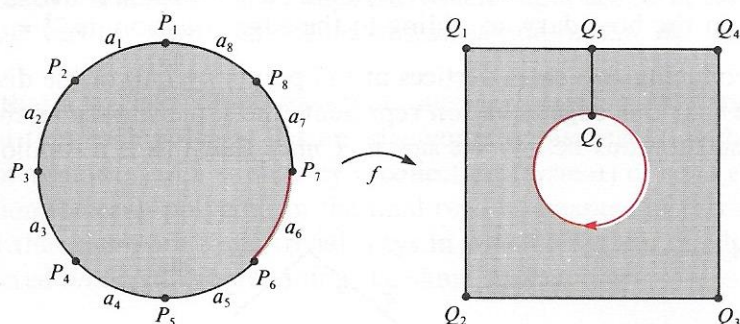
The conditions may be summarized as follows. The singular topological polygon π is the image of D under a continuous function f which is one-to-one except possibly on the boundary of D . On the boundary of D , f is

allowed to identify two edges to the same curved arc in π so long as not more than two points of the boundary (other than vertices) are mapped to the same point of π . We may have more than two vertices of D mapped to the same point of π . As an example, consider the composition of the two maps below.



Here, g is any homeomorphism of the disc with the rectangle which takes the vertices to the vertices, and h is the usual (identification) map from the rectangle to the torus. This exhibits the torus as a singular topological polygon. Each point on the boundary of D is identified with at least one other point — the vertices are all mapped to the same point (so $h \circ g$ is 4-to-1 on the vertices).

As a second illustration, consider the following.



The map f exhibits the square with a circular hole in it as a singular topological octagon. f maps P_1P_2 to Q_1Q_2 , P_2P_3 to Q_2Q_3 , ..., P_5P_6 to Q_5Q_6 ; then P_6P_7 is mapped round the circle in the direction given by the arrow, P_7P_8 maps to Q_6Q_5 , and finally P_8P_1 maps to Q_5Q_1 . Notice that, although a_6 is not identified with any other edge, P_6 and P_7 are both mapped to Q_6 — this illustrates part of condition (3a). ▲

In practice, we shall usually represent singular polygons not by maps of the disc, as above, but by polygons whose edges are labelled to show any identifications (as we have done up to now).

EXAMPLE 1 B page 82, Question 5

Solution

The function defined on the disc D by

$$f(x, y) = (|x|, y, (x^2 + y^2 - 1)x)$$

is certainly continuous, since each of the three coordinate functions

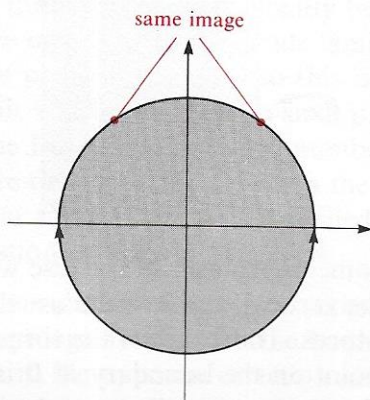
$$(x, y) \mapsto |x|, \quad (x, y) \mapsto y, \quad (x, y) \mapsto (x^2 + y^2 - 1)x$$

is continuous. To check that it is a polygon, we must first check that it is one-to-one except possibly on the boundary. So suppose that $f(x_1, y_1) = f(x_2, y_2)$, so that

$$(|x_1|, y_1, (x_1^2 + y_1^2 - 1)x_1) = (|x_2|, y_2, (x_2^2 + y_2^2 - 1)x_2).$$

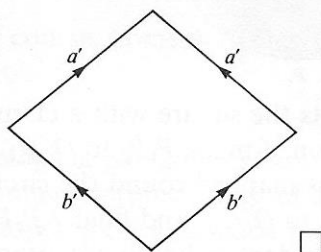
Comparing the first coordinates, $x_1 = \pm x_2$, and from the second coordinates, $y_1 = y_2$. Therefore $x_1^2 + y_1^2 - 1 = x_2^2 + y_2^2 - 1$. Unless this expression is zero —

in which case (x_1, y_1) and (x_2, y_2) represent points on the unit circle (the boundary of D) — we can conclude from the third coordinates that $x_1 = x_2$. But on the boundary we cannot make this final deduction and are left with the equations $x_1 = \pm x_2, y_1 = y_2$. Hence on the boundary, $f(x, y) = f(-x, y)$, and any point on the right-hand semicircle has the same image as its reflection in the y -axis.



We can, therefore, see that the image of f is obtained from D by identifying points on the boundary according to the edge equation $aa^{-1} = 1$.

By introducing two extra vertices at the points $(\pm 1, 0)$ of the disc boundary, we can see that this edge equation represents the same surface as that obtained from the rhombus below. We saw in *Unit 1* that this is a topological sphere.



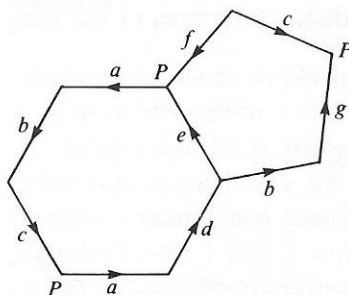
The next reading passage tells us how to recognize that a union of polygons is a surface, and gives conditions for this to be true. In *Units 1* and *2* we have subdivided surfaces into polygons which fit together so as to satisfy these conditions: from now on we shall assume that surfaces are given by specifying a number of polygons and how they fit together. **B** first explains some ways in which we do *not* want to fit polygons together, and then formulates the conditions that have to be imposed to prevent these “misfits”.

READ B page 56 to page 59, line 6.

NOTE Page 58, line –10 (Definition)

From our point of view, this definition is really a theorem: any set of points satisfying these conditions is, in fact, a surface according to the definition in *Unit 1*. The condition that there are only finitely many polygons ensures that the surface is compact (closed and bounded), while condition 4 says it is path connected.

To illustrate condition 2 of the definition, consider the following surface which is given as the union of two (singular) polygons.



The edge symbols (with arrows to show the direction in which they are to be identified) are as shown. The identification of *edges* tells us how *vertices* are to be identified. For example, the vertex marked *P* at the right of the diagram is at the “sharp” end of *c* so it must also be identified with the vertex at the bottom left when the edges *c* are identified. Similarly, when the edges *a* are identified we can see that *P* occurs in the third position indicated.

In fact, the nine vertices shown in the diagram, after the edge identifications are made, become just three distinct vertices on the surface (see SAQ 1). The meaning of condition 2 is that the identification of vertices is exactly that determined by the edge identifications — there are no extra vertex identifications that cannot be deduced from the edge identifications by arguing as we did above. Therefore, if we stipulate which edges are to be identified on a polygon, the vertices look after themselves, by condition 2. ▲

Knowing that a surface comprises a finite number of polygons, appropriately joined, and that each polygon has an edge equation associated with it, we now have a representation of a surface by a collection (system) of edge equations, one equation for each polygon. In the final reading passage of this section, **B** points out that there are some trivial ways in which a system of edge equations can be altered while still representing the same surface.

READ B page 59, line 7 to the end of Section 2.1.

EXAMPLE 2 Show that the system of edge equations

$$abac = 1; \quad cedb = 1; \quad e^{-1}fd^{-1} = 1$$

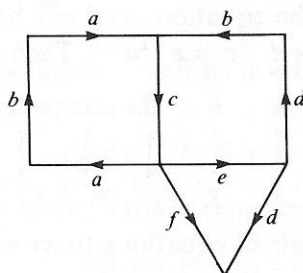
represents a surface.

Solution

The conditions 1 and 2 of **B** page 58 hold automatically when we specify a surface by means of edge equations. Condition 3 is easily checked, and no edge occurs more than twice in the equations.

Finally, the fact that *b* occurs in each of the first two equations shows that the polygons they represent are joined, and the occurrence of *e* (or e^{-1}) in the second two equations shows that the third polygon is joined to the second. Hence the polygons cannot be divided into two sets with no common edge.

We can illustrate this by drawing the polygons and joining them physically on identified edges.

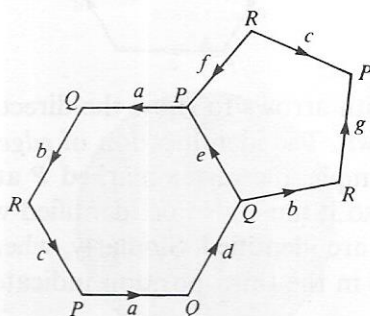


SAQ 1 Determine the set of distinct vertices in the diagram illustrating the surface at the top of page 9.

SAQ 2 B page 81, Question 1.

SAQ 3 B page 82, Question 3.

**SOLUTION
TO SAQ 1**



**SOLUTION
TO SAQ 2**

- (a) is not a surface because edge b occurs three times.
 (b) is not a surface (at least, not a connected surface) since the second pair of equations have no letters in common with the first pair. (Each pair of equations does represent a surface, however.)
 (c) Each edge occurs exactly twice and this represents a surface without boundary (closed surface).
 (d) is a surface, but d is a boundary edge since it occurs only once. ■

**SOLUTION
TO SAQ 3**

- (a) By a sequence of transformations, $ab^{-1}cd^{-1} = 1$ becomes $b^{-1}cd^{-1} = a^{-1}$, then $cd^{-1} = ba^{-1}$, and finally $c = ba^{-1}d$. Similarly, leaving e on the left and transporting the other three edge symbols to the right, $a^{-1}ecb = 1$ becomes $e = ab^{-1}c^{-1}$.
 (b) This equivalence is achieved by renaming the edges. One way to start is to re-label edge c in the left-hand equation as x , giving the new (but equivalent) equation $aba^{-1}xd^{-1}ebxf = 1$. As c no longer appears, we can now rename a^{-1} as c (so that a becomes c^{-1}), giving the equivalent form $c^{-1}bcxd^{-1}ebxf = 1$. Continuing in this way we might proceed as follows.

$$\begin{aligned}
 c^{-1}bcxd^{-1}ebxf = 1 &\longrightarrow c^{-1}bcxd^{-1}ybf = 1 && \text{(renaming } e \text{ as } y) \\
 &\longrightarrow c^{-1}bce^{-1}d^{-1}ybe^{-1}f = 1 && \text{(renaming } x \text{ as } e^{-1}) \\
 &\longrightarrow c^{-1}bce^{-1}a^{-1}ybe^{-1}f = 1 && \text{(renaming } d \text{ as } a) \\
 &\longrightarrow c^{-1}dce^{-1}a^{-1}yde^{-1}f = 1 && \text{(renaming } b \text{ as } d) \\
 &\longrightarrow c^{-1}dce^{-1}a^{-1}yde^{-1}b = 1 && \text{(renaming } f \text{ as } b) \\
 &\longrightarrow c^{-1}dce^{-1}a^{-1}f^{-1}de^{-1}b = 1 && \text{(renaming } y \text{ as } f^{-1})
 \end{aligned}$$

We have done this by renaming letters one at a time, but we could have done it in one step if we had been prepared to use the simultaneous “renaming function”

$$\begin{array}{cccccc}
 a & b & c & d & e & f \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 c^{-1} & d & e^{-1} & a & f^{-1} & b
 \end{array}$$

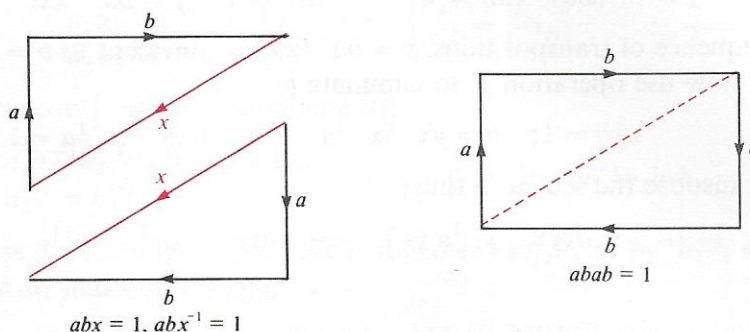
- (c) By transpositions, the equation $abcd = 1$ becomes $ab = d^{-1}c^{-1}$ and $aed^{-1}c = 1$ becomes $d^{-1}c = e^{-1}a^{-1}$. Then renaming by the prescription

$$\begin{array}{ccccc}
 a & b & d & c & e \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 x & y & z^{-1} & w^{-1} & t^{-1}
 \end{array}$$

changes the latter pair of equations to $xy = zw$ and $zw^{-1} = tx^{-1}$, as required. ■

3.2 Cut and Paste Operations

We know how to represent a surface combinatorially by a system of edge equations, which we can think of as arising from a subdivision of the surface. We now have to discover what changes can be legitimately made in this system of equations without changing the topological type (or homeomorphism class) of the surface it represents. We have already seen that some changes are permissible, namely the trivial equivalences 1 and 2 summed up on **B** page 61. We can spot other equivalences very easily by thinking about what happens if we cut up a surface into smaller polygons, or paste polygons together.



Consider the surface given by the pair of edge equations $abx = 1$, $abx^{-1} = 1$. Drawing the corresponding triangles, it is evident that when we glue them along the edge x we obtain the rectangle with edge equation $abab = 1$, and that this has not changed the homeomorphism class. Conversely, we could have cut the rectangle along the dotted line to obtain the two triangles.

This process “paste two polygons together” (along a common edge) — and its inverse “cut a polygon in two” — is one of three basic geometric “cut and paste” operations we can perform on polygons. It turns out that if the two sets of polygons represent a subdivision of the same surface, then we can convert one set of polygons into the other by a sequence of cut and paste operations. In order to prove this, it is necessary to work algebraically with the edge equations, so the first task is to see what the “cut and paste” operations can do to a system of edge equations.

READ B Section 2.2 on page 61 to page 64, line –10.

NOTE Page 64, line 18

B's statement that “the reflexive law is generally false” is a subtle point. He is *not* claiming that in some sense $a \neq a$. However, the equation $a = a$ cannot occur as one of several in a system of edge equations for a surface. This is because no edge can occur more than twice; so if $a = a$ is an equation, a cannot occur in any other equation of the system. But then, if there is another equation, the surface is not connected (condition 4 on **B** page 58, or see (b) of the definition on **B** page 61). So $a = a$ would be the *only* equation in the system (and the surface is a sphere). ▲

EXAMPLE 3 Using the “cut and paste” operation 1, and its inverse, show that the surface represented by the single edge equation

$$aba^{-1}cb^{-1}c = 1$$

is combinatorially equivalent to the surface represented by

$$aabbcc = 1.$$

Solution

Using operation 1 we can replace cb^{-1} by the new letter x and obtain the two equations

$$aba^{-1}xc = 1; \quad x = cb^{-1}.$$

Now, by a trivial equivalence (**B** page 61), the equation $x = cb^{-1}$ becomes $xb = c$ and we can now use operation 1' to eliminate the letter c :

$$aba^{-1}xc = 1; \quad c = xb \longrightarrow aba^{-1}xxb = 1.$$

We now use operation 1 again. Write $y = ba^{-1}xx$:

$$aba^{-1}xxb = 1 \longrightarrow ayb = 1; \quad y = ba^{-1}xx.$$

By a sequence of transpositions, $y = ba^{-1}xx$ is equivalent to $b = yx^{-1}x^{-1}a$, so we can now use operation 1' to eliminate b :

$$ayb = 1; \quad b = yx^{-1}x^{-1}a \longrightarrow ayyx^{-1}x^{-1}a = 1.$$

Now transpose the second a thus:

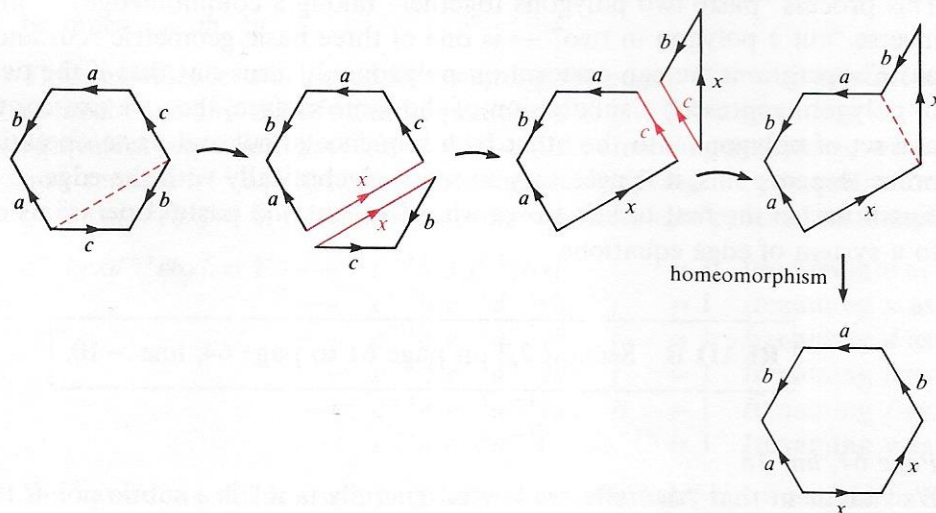
$$\begin{aligned} ayyx^{-1}x^{-1}a = 1 &\longrightarrow ayyx^{-1}x^{-1} = a^{-1} \\ &\longrightarrow aayyx^{-1}x^{-1} = 1. \end{aligned}$$

Finally, renaming y, x^{-1} by the "new" labels b, c we have

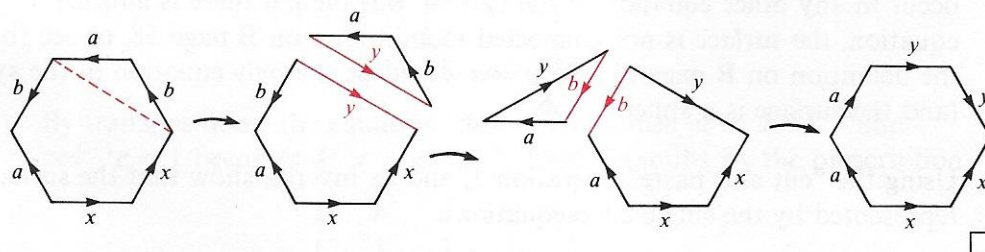
$$aabbcc = 1,$$

as required.

We can copy this algebraic proof with actual "cutting and pasting" along the following lines. The first use of 1 and 1' cuts the hexagon along the line x indicated and re-glues the two pieces along c .



Next we cut along y and re-glue along b .



The next example incorporates operation 3 and also illustrates the fact (to be proved in *Unit 4*) that any system of edge equations for a surface is equivalent to a system with only one equation.

EXAMPLE 4 B page 82, Question 3 on Section 2.2*Solution*

A glance at Fig. 1.26 on B page 28, where this system of edge equations originated, suggests that we can obtain a single polygon from the three given quadrilaterals by glueing the a_2 edges and the a_3 edges. Glueing the a_2 edges is achieved (algebraically speaking) by using operation 1' to eliminate a_2 . From the second of the given equations, $a_2 b_2^{-1} a_3^{-1} b_2 = 1$, a transposition gives $a_2^{-1} = b_2^{-1} a_3^{-1} b_2$. Applying 1' reduces the three equations to two:

$$\left. \begin{aligned} a_1 b_1^{-1} a_2^{-1} b_1 &= 1 \\ a_2 b_2^{-1} a_3^{-1} b_2 &= 1 \\ a_3 b_3^{-1} a_1^{-1} b_3 &= 1 \end{aligned} \right\} \longrightarrow \begin{cases} a_1 b_1^{-1} b_2^{-1} a_3^{-1} b_2 b_1 = 1 \\ a_3 b_3^{-1} a_1^{-1} b_3 = 1 \end{cases}$$

We can now use 1' again to eliminate a_3 :

$$\left. \begin{aligned} a_1 b_1^{-1} b_2^{-1} a_3^{-1} b_2 b_1 &= 1 \\ a_3^{-1} &= b_3^{-1} a_1^{-1} b_3 \end{aligned} \right\} \longrightarrow a_1 b_1^{-1} b_2^{-1} b_3^{-1} a_1^{-1} b_3 b_2 b_1 = 1.$$

We now use 3'. The edges b_1, b_2 occur together as $b_2 b_1$ or $b_1^{-1} b_2^{-1}$, so we can write $x = b_2 b_1$ and obtain

$$a_1 b_1^{-1} b_2^{-1} b_3^{-1} a_1^{-1} b_3 b_2 b_1 = 1 \longrightarrow a_1 x^{-1} b_3^{-1} a_1^{-1} b_3 x = 1.$$

We can use 3' again since b_3, x occur together in the correct way; write $y = b_3 x$ and then

$$a_1 x^{-1} b_3^{-1} a_1^{-1} b_3 x = 1 \longrightarrow a_1 y^{-1} a_1^{-1} y = 1.$$

Finally, the trivial equivalence of renaming yields the required equation:

$$a b a^{-1} b^{-1} = 1. \quad \square$$

After a little practice, one can make these transformations quickly and automatically; the point about the notation is that one can perform algebraic manipulations on the equations following a few simple rules that look very similar to ordinary rules of algebra. (We shall develop this further in *Unit 4*.)

SAQ 4 B page 82, Question 4 on Section 2.2.

SAQ 5 B page 83, Question 6 on Section 2.2.

SAQ 6 Show that the surface with edge equation

$$a b c d e d^{-1} e c^{-1} a^{-1} f = 1$$

is combinatorially equivalent to the surface with equation

$$a a b b c d c^{-1} e f e^{-1} = 1.$$

SOLUTION TO SAQ 4 Figure 1.27 on B page 28 suggests that we eliminate a_3, a_4, b_3 and b_4 . Algebraically, this corresponds to repeated use of operation 1'.

By trivial equivalences, the second equation of the system can be written $a_3 = b_4^{-1} a_1 b_2$. Substituting this in the first equation we obtain $b_4^{-1} a_1 b_2 b_1 a_1^{-1} b_3^{-1} = 1$. In a similar way, the third equation is equivalent to $b_3^{-1} = a_2^{-1} b_1^{-1} a_4$, and substitution of this in the equation just obtained gives the system

$$b_4^{-1} a_1 b_2 b_1 a_1^{-1} a_2^{-1} b_1^{-1} a_4 = 1; \quad a_2 b_4 a_4^{-1} b_2^{-1} = 1.$$

From the second of these two equations, we obtain $a_4 = b_2^{-1} a_2 b_4$, which enables us to substitute for a_4 in the first and obtain the single equation

$$b_4^{-1} a_1 b_2 b_1 a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} a_2 b_4 = 1.$$

Now move the b_4 from the end to the beginning, and observe that the pair $b_4 b_4^{-1}$ cancel by rule 2' to give

$$a_1 b_2 b_1 a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} a_2 = 1.$$

Now move a_2 from the end to the beginning. Finally, using 3' and putting $x = b_2 b_1$ and $y = a_2 a_1$ yields $xyx^{-1}y^{-1} = 1$. After renaming letters, this is the required equation. ■

SOLUTION TO SAQ 5 Substituting $e_1^{-1} = d_2 a_2 b_1$ from II into I (operation 1') gives the single equation replacing I and II

$$b_1 b_2 a_1 d_1 d_2 a_2 = 1.$$

Similarly, substituting $e_2 = d_2 a_1^{-1} c_1$ from IV into III gives the single equation

$$c_1 c_2 a_2^{-1} d_1 d_2 a_1^{-1} = 1. \quad \blacksquare$$

SOLUTION TO SAQ 6 $abcded^{-1}ec^{-1}a^{-1}f = 1 \rightarrow abcdxeca^{-1}f = 1; \quad x = ed^{-1} \quad (\text{operation 1})$
 $\rightarrow abcdxxdc^{-1}a^{-1}f = 1 \quad (\text{operation 1', eliminating } e = xd)$
 $\rightarrow abcydc^{-1}a^{-1}f = 1; \quad y = dxx \quad (\text{operation 1})$
 $\rightarrow abcyx^{-1}x^{-1}c^{-1}a^{-1}f = 1 \quad (\text{operation 1', eliminating } d = yx^{-1}x^{-1})$
 $\rightarrow yyx^{-1}x^{-1}c^{-1}a^{-1}fabc = 1 \quad (\text{trivial})$
 $\rightarrow yyx^{-1}x^{-1}zfabc; \quad z = c^{-1}a^{-1} \quad (\text{operation 1})$
 $\rightarrow yyx^{-1}x^{-1}zfz^{-1}c^{-1}bc = 1 \quad (\text{operation 1', eliminating } a = z^{-1}c^{-1})$
 $\rightarrow aabbcdc^{-1}efe^{-1} = 1 \quad (\text{renaming}). \quad \blacksquare$

We conclude this section with two observations on how certain *algebraic* facts about the edge equations of a surface correspond to *geometric* properties. The first observation was actually made in SAQ 15 of Unit 2 (see page 60). If the equation, or any one equation, for a surface looks like

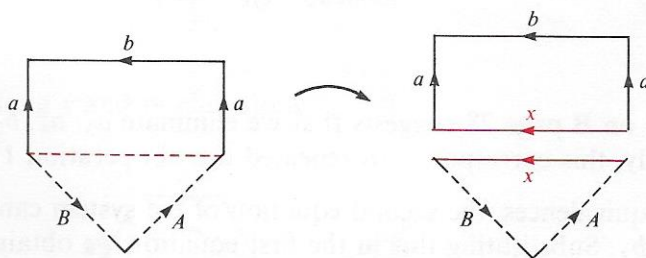
$$\dots a \dots a \dots = 1 \quad \text{or} \quad \dots a^{-1} \dots a^{-1} \dots = 1$$

with the letter a occurring twice in the same sense, then we can cut out a Möbius band from the surface and so *the surface is not orientable*.

A different geometric feature can be recognized when the equation has the form

$$Aaba^{-1}B = 1,$$

and b does not occur in A or B , namely that *the surface has a boundary circle b* . To see this, consider the following diagram.



Cutting off part of the surface along the dotted line (which, by operation 1, is equivalent algebraically to taking the two equations $AxB = 1$, $x = aba^{-1}$), the portion cut off is a cylinder and has the free boundary curve b (which forms a circle after the edges a are identified). It is not identified with any other edge since b does not occur a second time in the equation. (This does not mean that b is the *only* boundary circle. Indeed, if A or B contains a string dcd^{-1} , where c does not appear elsewhere, then c is another boundary circle.)

In *Unit 4* we shall see how to manipulate the edge equations for any surface so as to get them into a simple form, called the *canonical equation*, from which we can read off all the geometric features that we have considered so far.

SAQ 7 Show that the following representations of the Möbius band are combinatorially equivalent.

(i) M^2 : $abac = 1$.

(ii) M^2 : $eef = 1$.

SAQ 8 Show that the following representations of the real projective plane are combinatorially equivalent.

(i) $\mathbb{RP}^2$: $abab = 1$.

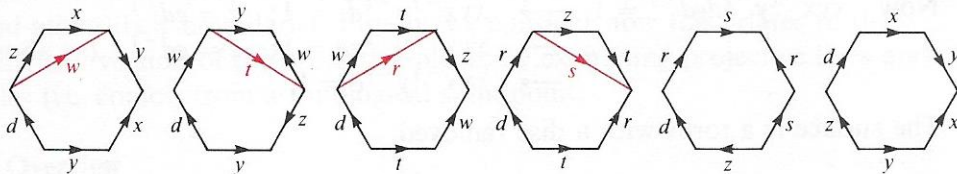
(ii) $\mathbb{RP}^2$: $ccdd^{-1} = 1$.

SAQ 9 Show that the representation

$$axbx^{-1} = 1; \quad b = 1$$

of the disc is combinatorially equivalent to the representation $a = 1$. Interpret your answer geometrically.

SAQ 10 The following sequence of diagrams illustrates a sequence of “cut and paste” operations on a topological hexagon.



Write out the combinatorial equivalences algebraically, and deduce that the following two representations are combinatorially equivalent:

(i) $xyx^{-1}y^{-1}dz = 1$; (ii) $xyx^{-1}y^{-1}zd = 1$.

Hence deduce that the representations

$$xyx^{-1}y^{-1}e = 1, \quad xyx^{-1}y^{-1}ded^{-1} = 1$$

are combinatorially equivalent. Interpret your answer geometrically.

**SOLUTION
TO SAQ 7**

$$\begin{aligned} abac = 1 &\longrightarrow xac = 1; \quad x = ab \\ &\longrightarrow xac = 1; \quad a = xb^{-1} \\ &\longrightarrow x(xb^{-1})c = 1 \\ &\longrightarrow eef = 1 \quad (\text{renaming } x \text{ as } e \text{ and writing } f \text{ for } b^{-1}c) \quad \blacksquare \end{aligned}$$

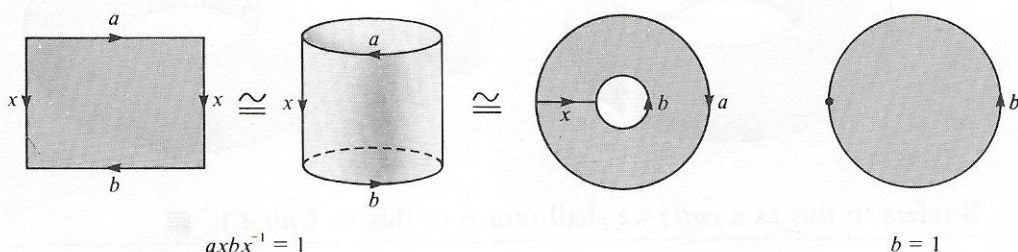
**SOLUTION
TO SAQ 8**

$$\begin{aligned} abab = 1 &\longrightarrow cc = 1 \quad (\text{writing } c \text{ for } ab) \\ &\longrightarrow ccdd^{-1} = 1 \quad \blacksquare \end{aligned}$$

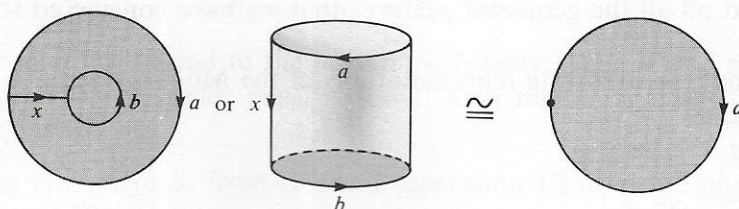
**SOLUTION
TO SAQ 9**

$$\begin{aligned} axbx^{-1} = 1; \quad b = 1 &\longrightarrow axx^{-1} = 1 \\ &\longrightarrow a = 1 \end{aligned}$$

$axbx^{-1} = 1$ is a cylinder (or annulus), whereas $b = 1$ is, of course, a disc.



Substituting for b amounts to glueing the two surfaces along the common boundary curve b , and the result is just the disc $a = 1$.



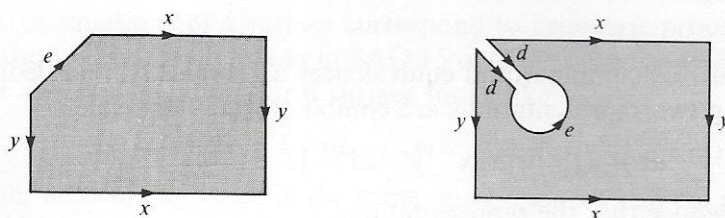
SOLUTION TO SAQ 10

$$\begin{aligned}
 xyx^{-1}y^{-1}dz = 1 &\longrightarrow zxyx^{-1}y^{-1}d = 1 \\
 &\longrightarrow wyx^{-1}y^{-1}d = 1; \quad w = zx \quad (\text{or } x^{-1} = w^{-1}z) \\
 &\longrightarrow wyw^{-1}zy^{-1}d = 1 \\
 &\longrightarrow wtzy^{-1}d = 1; \quad t = yw^{-1} \quad (\text{or } y^{-1} = w^{-1}t^{-1}) \\
 &\longrightarrow wtzw^{-1}t^{-1}d = 1 \\
 &\longrightarrow rzw^{-1}t^{-1}d = 1; \quad r = wt \quad (\text{or } w^{-1} = tr^{-1}) \\
 &\longrightarrow rztr^{-1}t^{-1}d = 1 \\
 &\longrightarrow rsr^{-1}t^{-1}d = 1; \quad s = zt \quad (\text{or } t^{-1} = s^{-1}z) \\
 &\longrightarrow rsr^{-1}s^{-1}zd = 1
 \end{aligned}$$

and renaming x and y gives the result. (Note that d and z are *unchanged* throughout. This is the objective, and so we cannot simply rename d and z , and vice versa, at the outset.)

$$\begin{aligned}
 \text{Now } xyx^{-1}y^{-1}ded^{-1} = 1 &\longrightarrow xyx^{-1}y^{-1}dz = 1; \quad z = ed^{-1} \\
 &\longrightarrow xyx^{-1}y^{-1}zd = 1; \quad z = ed^{-1} \quad (\text{as above}) \\
 &\longrightarrow xyx^{-1}y^{-1}e = 1.
 \end{aligned}$$

The surface is a torus with a disc removed.



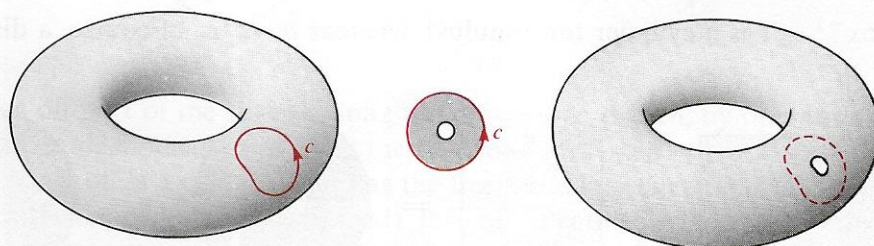
(The equivalence

$$\dots e = 1 \longrightarrow \dots ded^{-1} = 1,$$

where e is a boundary circle, can be thought of as another geometric operation, as follows. Replacing ded^{-1} by c in the second equation yields the combinatorially equivalent system

$$\dots c = 1; \quad c = ded^{-1}.$$

The second of these represents an annulus with c as one of its boundary circles, and it is geometrically obvious that glueing an annulus to a boundary circle of a surface does not alter the homeomorphism type.



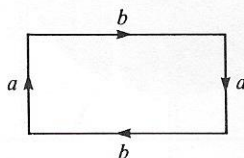
B refers to this as a *cuff*; we shall return to this in *Unit 4*.) ■

3.3 Descriptions of the Projective Plane

This section, which includes TV and TV-related audio, considers various geometric and topological descriptions of the projective plane.

NO PRE-TV WORK IS REQUIRED FOR WATCHING TV3 APART FROM READING THIS INTRODUCTION.

In *Unit 2* we represented the real projective plane as the surface obtained by identifying the two opposite pairs of edges of a rectangle, with edge equation $abab = 1$.



If, however, you have done any projective geometry (for example in *M101* or *M203*), you will have met something else called the “projective plane”. This is the plane in which we do projective geometry and at first sight does not look at all like the projective plane as defined above.

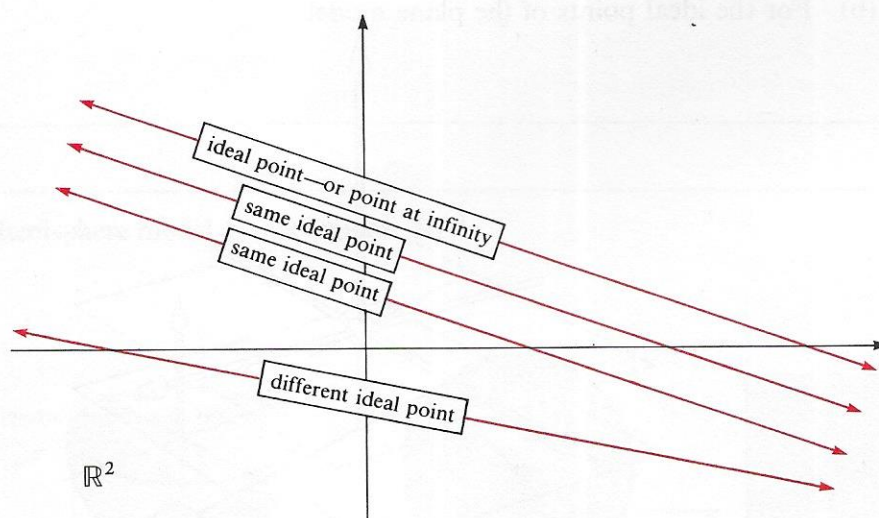
In TV3 we explain how the projective plane we meet in geometry is topologically equivalent to the surface defined above. In the second part of TV3, we show how the projective plane is made up of a Möbius band and a disc, glued along their boundaries. Finally, we consider how this relates to the geometer’s version of the projective plane, by examining projective lines and circles (i.e. conics) from a topological standpoint.

TV Overview

After watching TV3, look at Frame 1 and start TV-related Audio Cassette A1 at track 1, side 2.

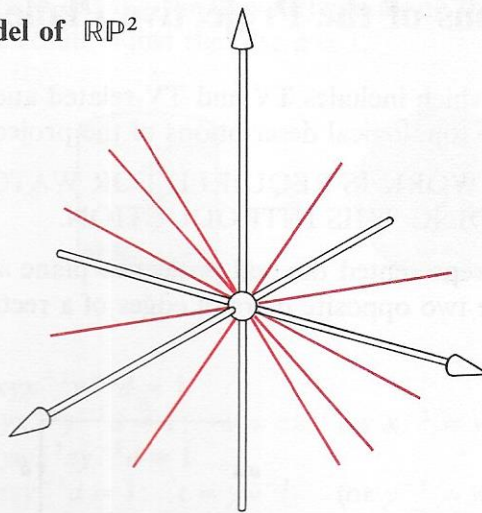
FRAME 1

Geometer’s “plane” model of $\mathbb{RP}^2$



FRAME 2

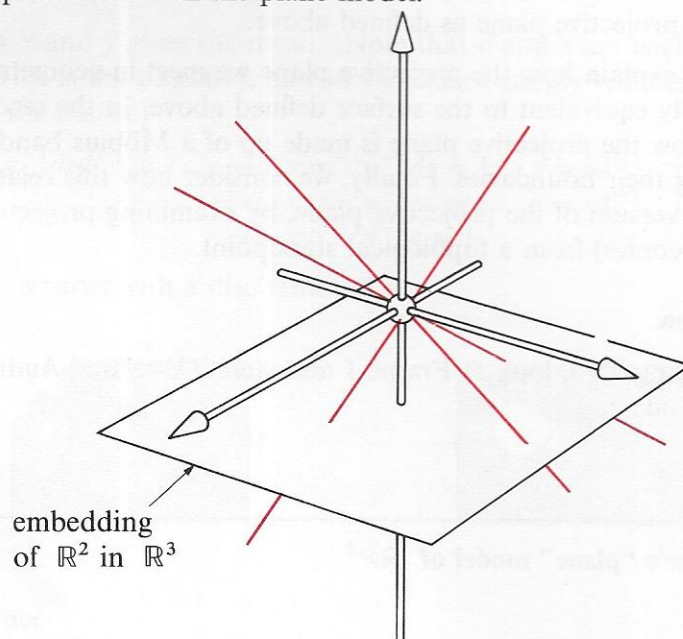
Geometer's "ray" model of $\mathbb{RP}^2$



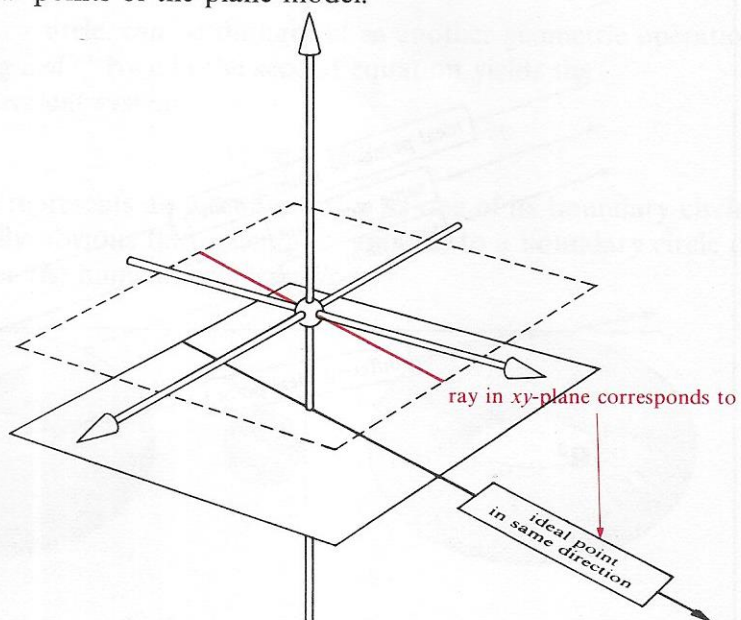
FRAME 3

Correspondence between "ray" and "plane" models

(a) For points of $\mathbb{R}^2$ in the plane model.



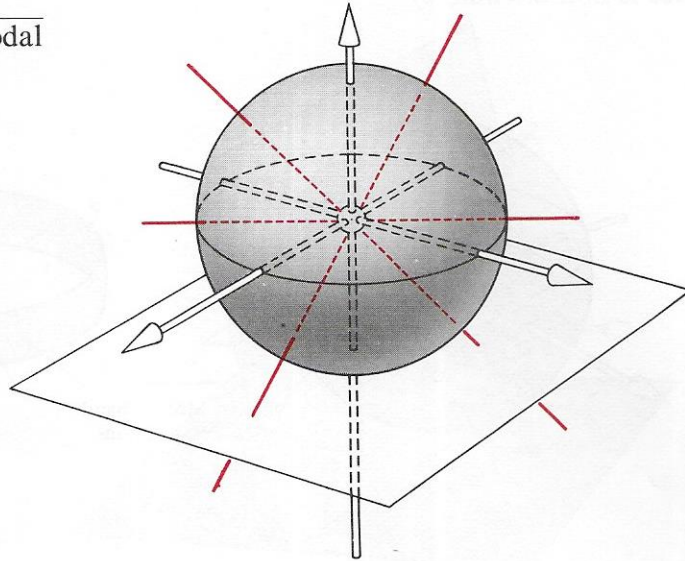
(b) For the ideal points of the plane model.



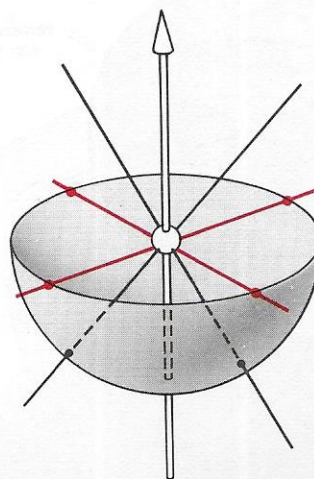
FRAME 4

Sphere model of $\mathbb{RP}^2$

$$\mathbb{RP}^2 \cong \frac{S^2}{\text{antipodal}}$$

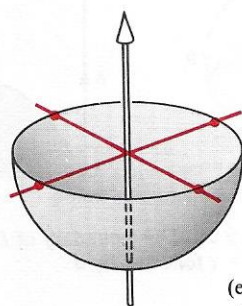


FRAME 5

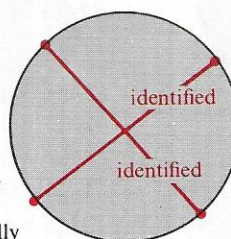
Hemisphere model of $\mathbb{RP}^2$ 

identify antipodal
pairs of points
on the equator

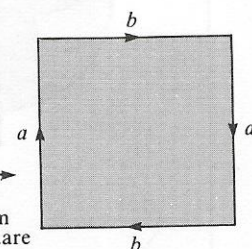
FRAME 6

Hemisphere model $\cong$ topologist's $\mathbb{RP}^2$ 

flatten
(e.g. vertically
downwards)
into a disc

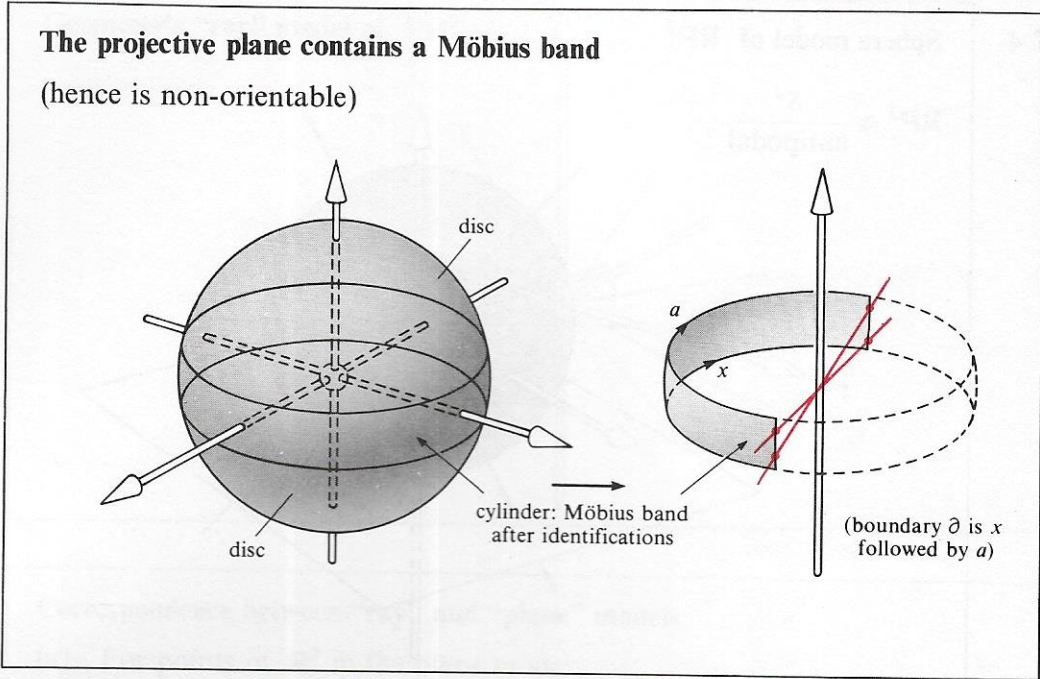


deform
into square



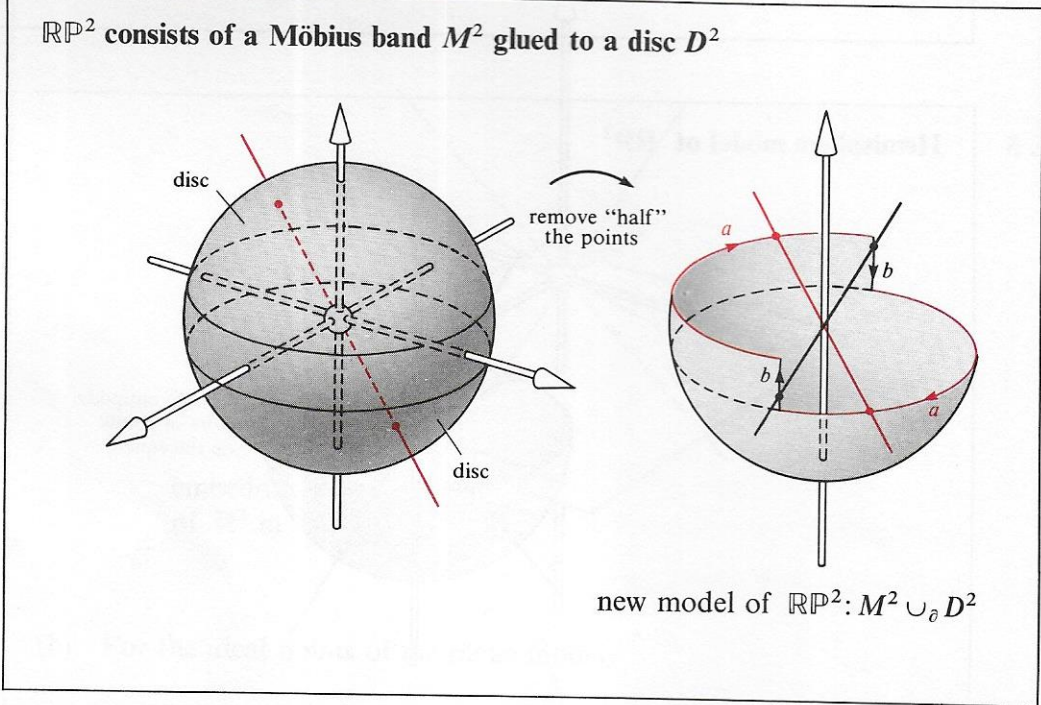
FRAME 7

The projective plane contains a Möbius band
(hence is non-orientable)



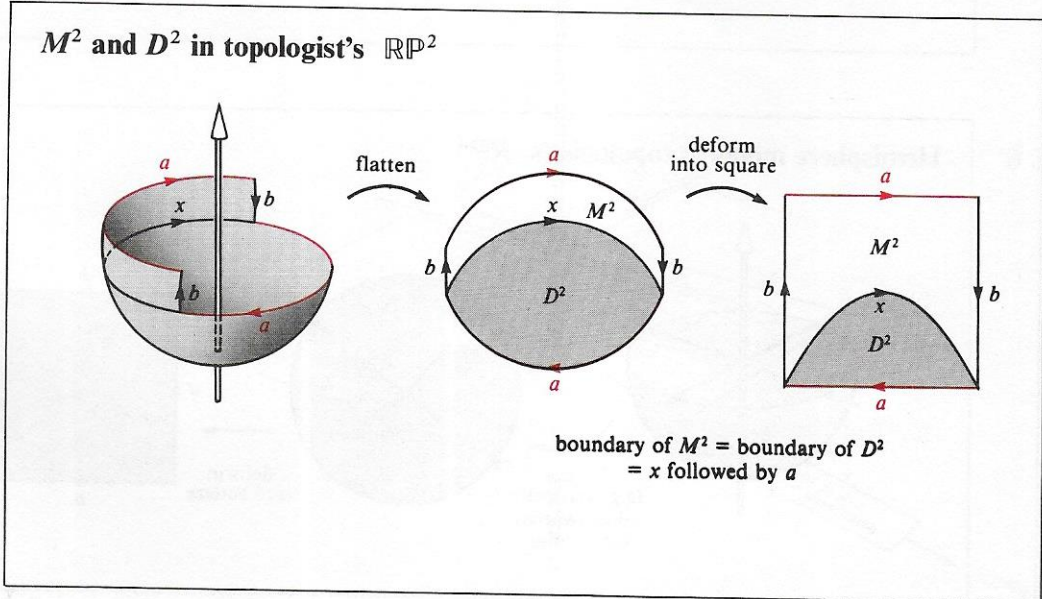
FRAME 8

$\mathbb{RP}^2$ consists of a Möbius band M^2 glued to a disc D^2



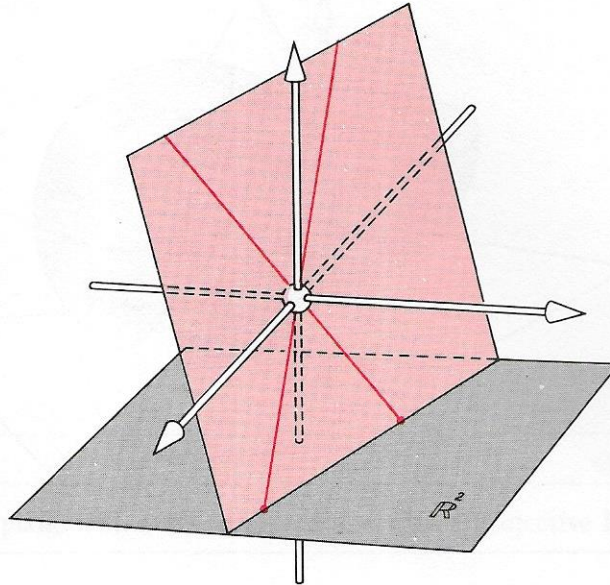
FRAME 9

M^2 and D^2 in topologist's $\mathbb{RP}^2$



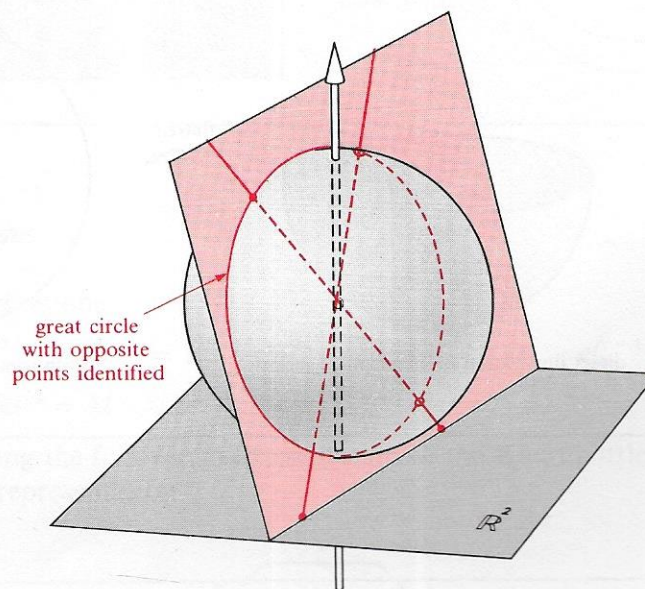
FRAME 10

Projective line in "plane" and "ray" models



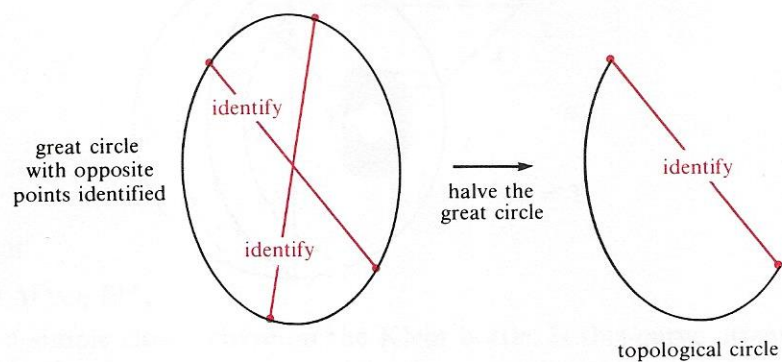
FRAME 11

Projective line in sphere model



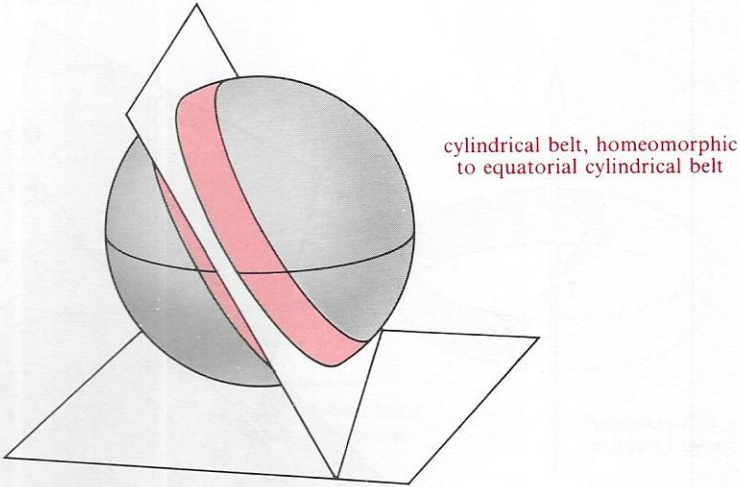
FRAME 12

A projective line is a topological circle



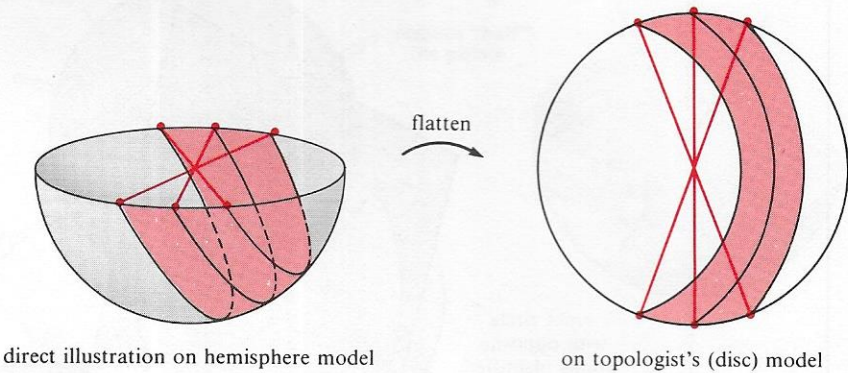
FRAME 13

Projective line is centre circle of a Möbius band



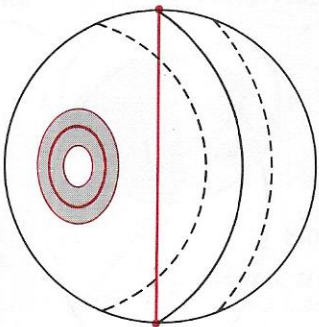
FRAME 14

Direct illustration of projective line as centre circle of Möbius band



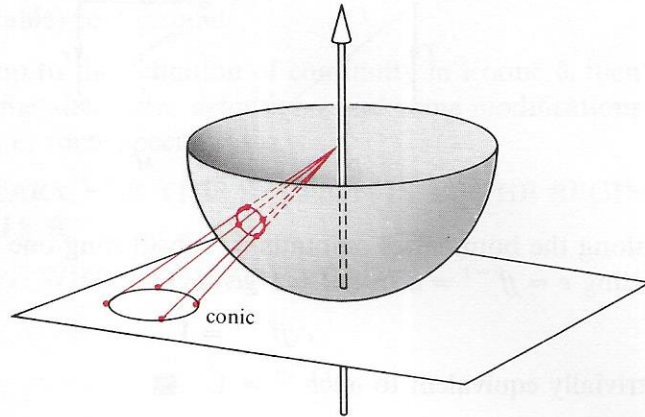
FRAME 15

A circle in $\mathbb{RP}^2$ which is NOT a projective line does NOT reverse orientation



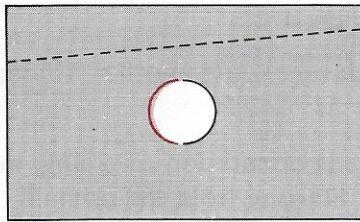
FRAME 16

A conic is NOT orientation-reversing

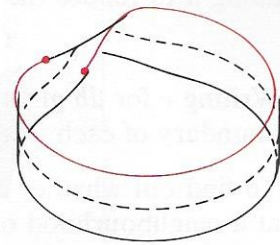


FRAME 17

The geometer's plane with a disc removed and with a projective line drawn on it



should be recognizable as a Möbius band with a centre circle drawn on it:



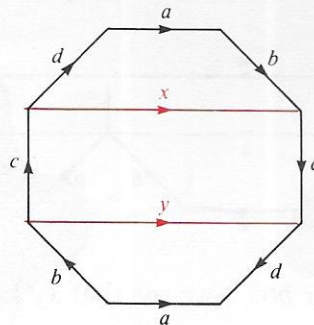
Post-TV Problems

SAQ 11 Use the representations

$$\mathbb{RP}^2: aabb^{-1} = 1, \quad M^2: ccd = 1, \quad D^2: eff^{-1} = 1$$

to verify that $\mathbb{RP}^2 = M^2 \cup_{\partial} D^2$.

SAQ 12 Show that cutting the following representation of the Klein bottle along the arcs x, y leads to a representation



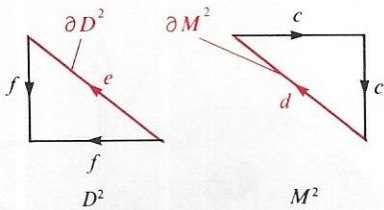
$$K^2: cxcy^{-1} = 1; \quad ex^{-1}ey = 1.$$

Deduce that

- (i) $K^2 = M^2 \cup_{\partial} M^2$;
- (ii) xy is a simple closed curve on the Klein bottle. Is this curve orientation-reversing?

**SOLUTION
TO SAQ 11**

The boundary curve of $D^2: eff^{-1} = 1$ is e , and of $M^2: ccd = 1$ is d .



Glueing along the boundaries amounts to substituting one boundary for the other. Putting $e = ff^{-1} = d$ in $ccd = 1$ gives

$$ccff^{-1} = 1,$$

which is trivially equivalent to $aabb^{-1} = 1$. ■

**SOLUTION
TO SAQ 12**

Cutting along x, y leads to the system of edge equations

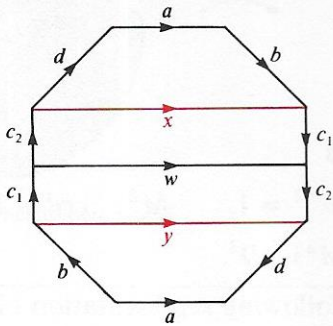
$$abx^{-1}d = 1; \quad xcy^{-1}c = 1; \quad yda^{-1}b = 1$$

for K^2 . The middle equation already represents a Möbius band, so we glue along a to reduce the other two equations to

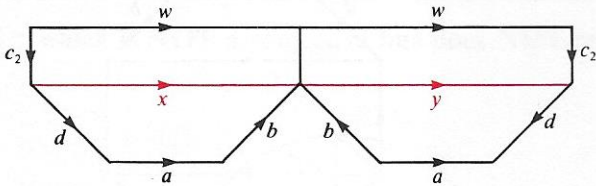
$$ydbx^{-1}db = 1.$$

Writing e for db gives $yex^{-1}e = 1$, which represents another Möbius band. The boundary of each is xy and hence $K^2 = M^2 \cup_{\partial} M^2$.

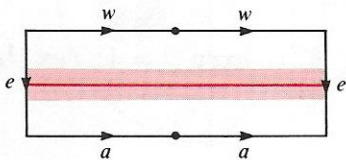
To find out whether or not the curve xy is orientation-reversing, we must look at a neighbourhood of the curve in K^2 . Starting from the original representation, the easiest way to do this is to cut along the new edge w and



glue along c_1 so as to assemble the curve xy as a single arc.



Eliminating the edge pair bb^{-1} , we see that xy has an annulus neighbourhood



and is thus orientation preserving! ■

3.4 Continuous Functions

This material *will not be assessed*. If you are short of time, you may treat it as optional (but desirable) background.

The section leads up to the definition of continuity in Frame 6, then looks at some examples, some alternative definitions, and some modifications of the definition required in some special cases.

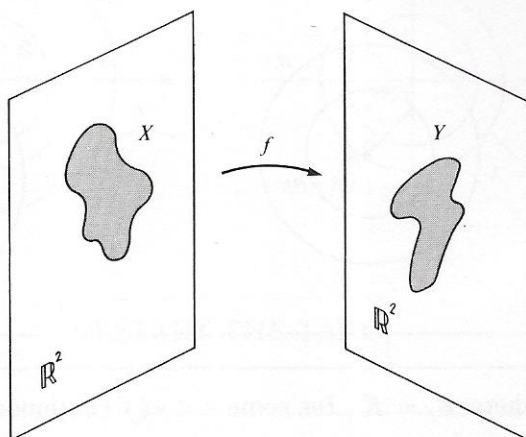
THE COMMENTARY FOR THIS SECTION IS AT THE BEGINNING OF AUDIO CASSETTE B.

START THE TAPE WHEN YOU ARE READY.



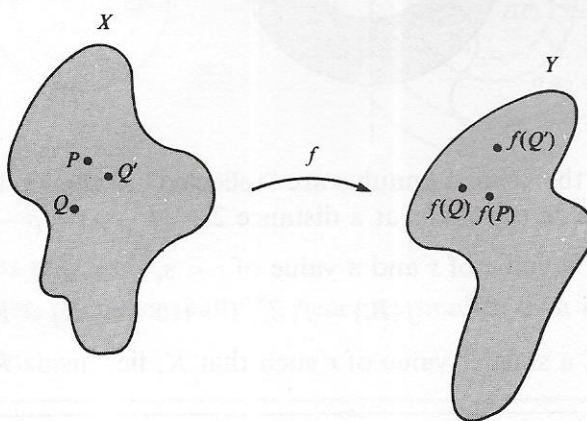
FRAME 1

Continuous functions on subsets of $\mathbb{R}^2$



FRAME 2

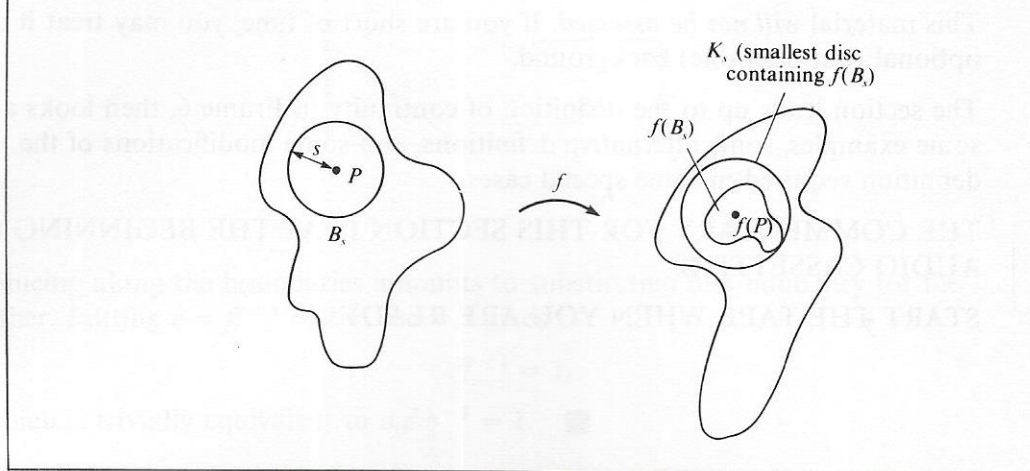
Informal description of continuity at P



f continuous at P : maps points near P to points near $f(P)$

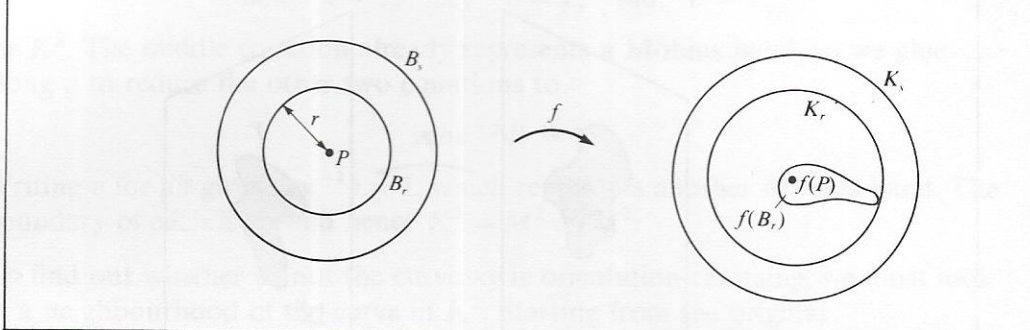
FRAME 3

Basic idea needed for rigorous definition of continuity



FRAME 4

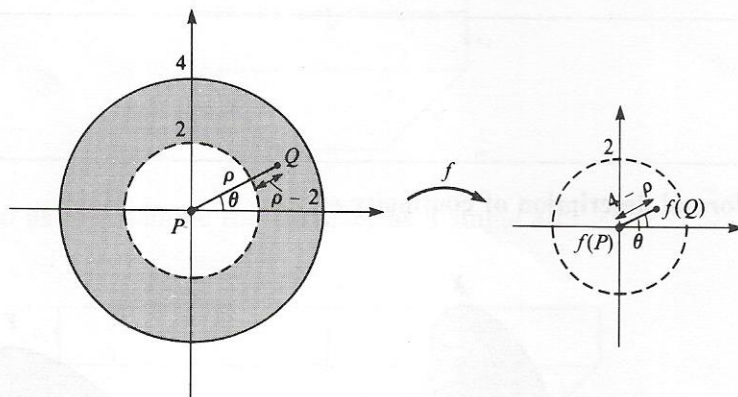
A smaller disc in the domain



FRAME 5

Example where $K_r = K_s$, for some $r < s$ (f continuous)

$$f(\rho, \theta) = \begin{cases} (\rho, \theta) & \text{for } \rho \leq 2 \\ (4 - \rho, \theta) & \text{for } 2 < \rho \leq 4 \text{ (shaded)} \\ \text{origin} & \text{for } \rho > 4 \end{cases}$$



Points in the shaded annulus are “reflected” in the circle of radius 2, i.e. $f(Q)$ is inside the circle, at a distance $2 - (4 - \rho) = \rho - 2$ from the circle.

- (i) Find a value of s and a value of $r < s$, such that

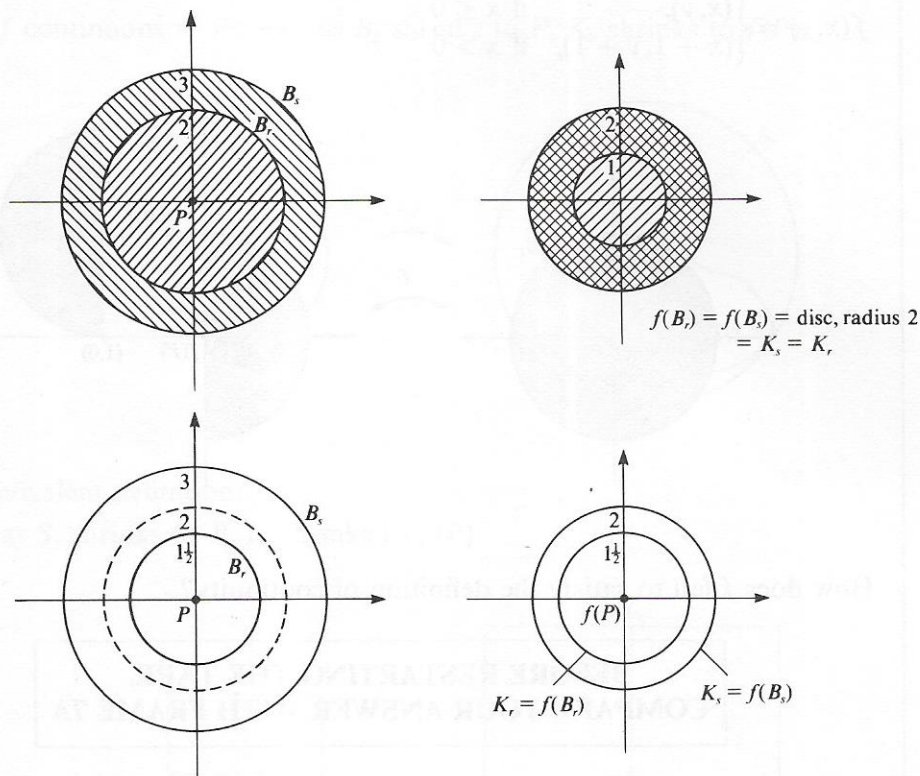
$$f(B_r) = f(B_s) \quad (\text{so that } K_r = K_s).$$

- (ii) Find a smaller value of r such that K_r lies inside K_s .

**BEFORE RESTARTING THE TAPE,
COMPARE YOUR ANSWERS WITH FRAME 5A**

FRAME 5A

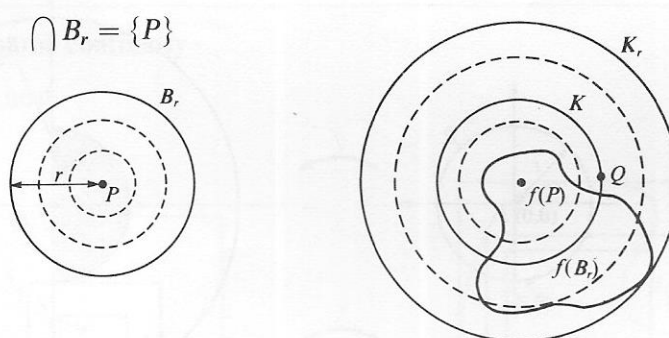
Solutions for Frame 5



RESTART THE TAPE

FRAME 6

Definition of continuity



For f continuous at P :

as r gets smaller, then eventually K_r becomes smaller than *any* K .

Formal definition:

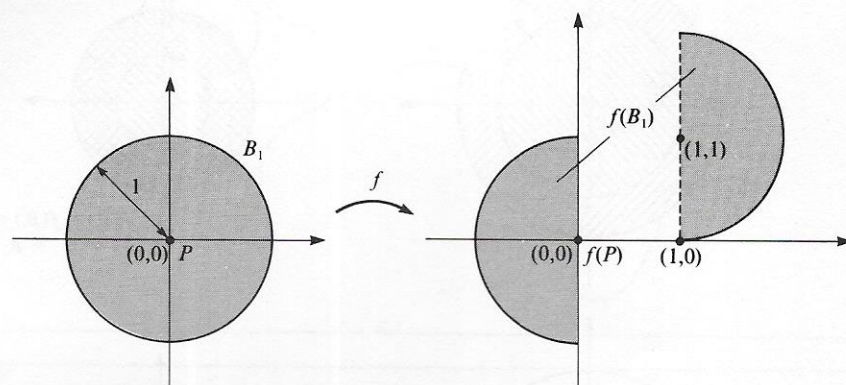
$$\text{for } \bigcap B_r = \{P\}, f \text{ continuous at } P \Leftrightarrow \bigcap K_r = \{f(P)\}$$

(i.e. as r shrinks to zero, the radius of K_r shrinks to zero).

FRAME 7

Intuitively obvious discontinuous function

$$f(x, y) = \begin{cases} (x, y), & \text{if } x \leq 0 \\ (x + 1, y + 1), & \text{if } x > 0 \end{cases}$$



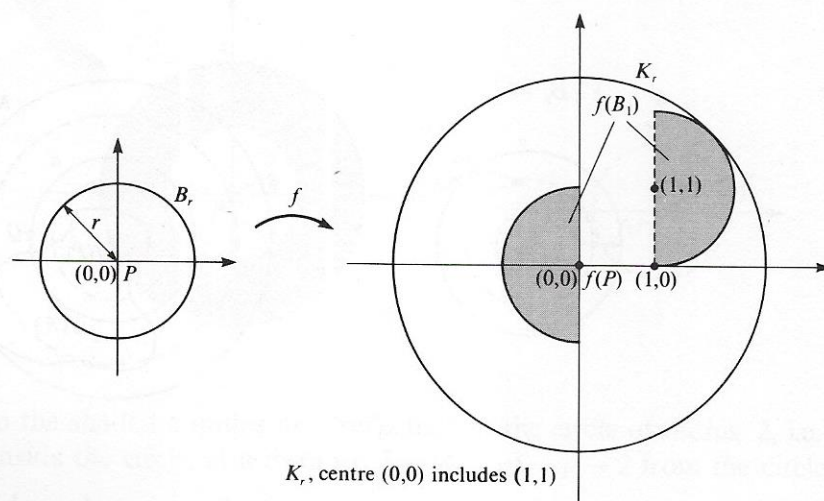
How does f fail to satisfy the definition of continuity?

**BEFORE RESTARTING THE TAPE,
COMPARE YOUR ANSWER WITH FRAME 7A**

FRAME 7A

Solution for Frame 7

$$f(x, y) = \begin{cases} (x, y), & \text{if } x \leq 0 \\ (x + 1, y + 1), & \text{if } x > 0 \end{cases}$$



$$\bigcap K_r = \text{disc of radius } \sqrt{2}$$

therefore $\bigcap K_r \neq \{f(P)\} \Rightarrow f$ not continuous at P .

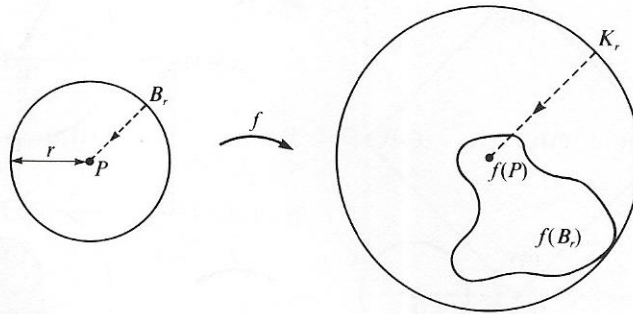
RESTART THE TAPE

FRAME 8

Equivalent definition of continuity

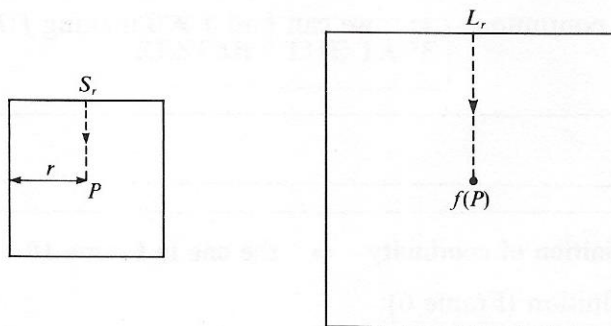
Our (Frame 6) definition:

f continuous at $P \Leftrightarrow$ as B_r shrinks to P , K_r shrinks to $f(P)$.



Equivalent definition:

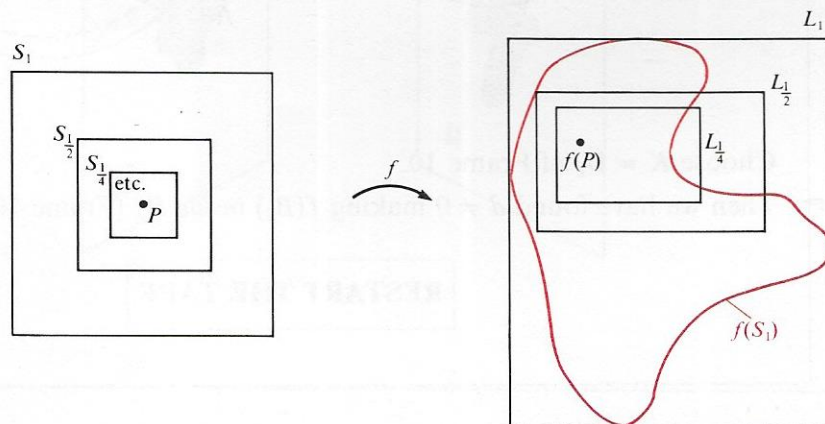
as S_r shrinks to P , L_r shrinks to $f(P)$.



FRAME 9

M203 definition of continuity

Standard test nest

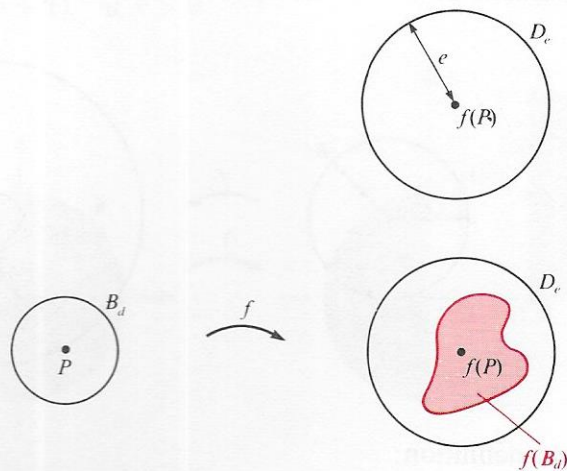


$$f \text{ continuous at } P \Leftrightarrow \bigcap L_{2^{-r}} = \{f(P)\}$$

FRAME 10

Another definition of continuity

Start with disc in *codomain*.



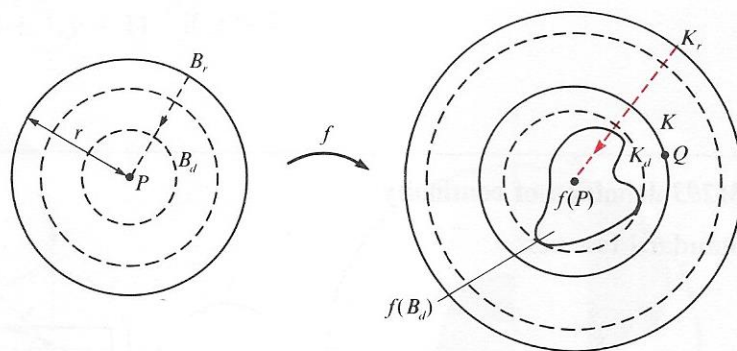
f continuous $\Leftrightarrow$ we can find $d \neq 0$ making $f(B_d)$ inside D_e .

FRAME 11

Our definition of continuity $\Rightarrow$ the one in Frame 10

Our definition (Frame 6):

as r gets smaller, then eventually K_r will get smaller than *any* disc K .



Choose $K = D_e$ of Frame 10.

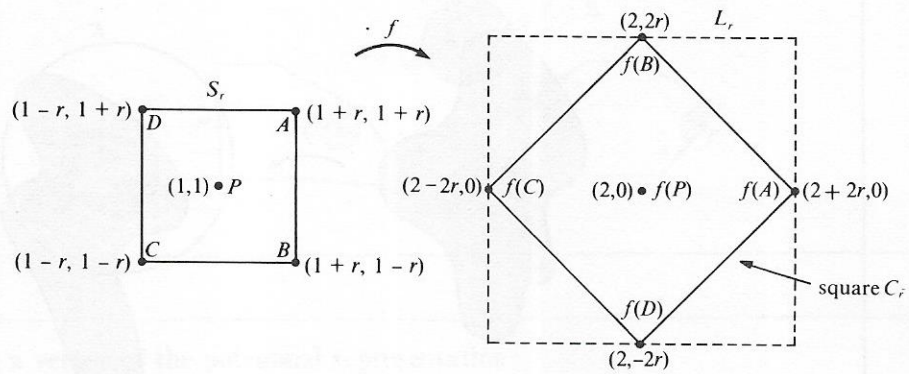
Then we have found $d \neq 0$ making $f(B_d)$ inside D_e (Frame 10 condition).

RESTART THE TAPE

FRAME 12

Checking continuity: a specific example

$$f(x, y) = (x + y, x - y)$$

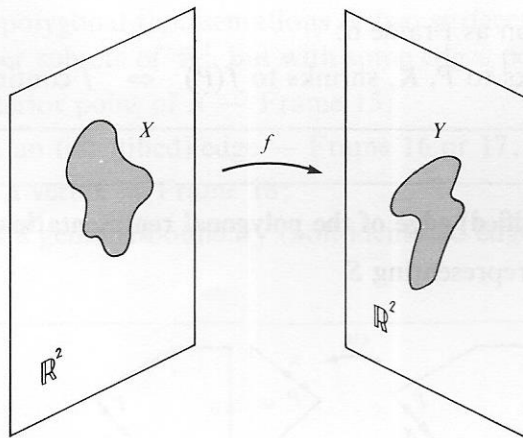


Confirm that the centre and vertices of S_r map to the centre and vertices of C_r . Hence show that L_r shrinks to zero as S_r does.

RESTART THE TAPE

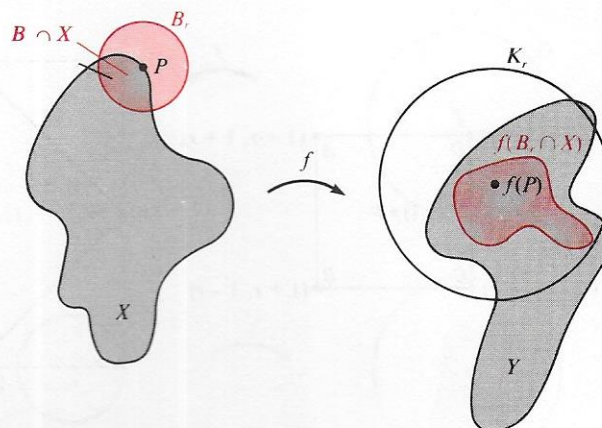
FRAME 13

X, Y proper subsets of $\mathbb{R}^2$



FRAME 14

Modified definition to take account of X and Y proper subsets of $\mathbb{R}^2$



Same definition as Frame 6:

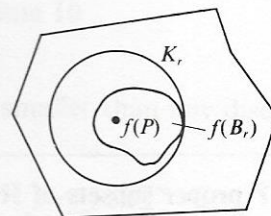
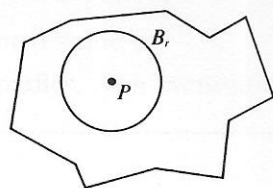
as B_r shrinks to P , K_r shrinks to $f(P) \Leftrightarrow f$ continuous at P .

FRAME 15

Continuity for an INTERIOR point P of a polygonal representation

polygon X representing S

polygon Y representing T



edge equation $A = 1$

edge equation $B = 1$

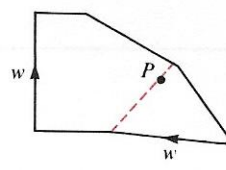
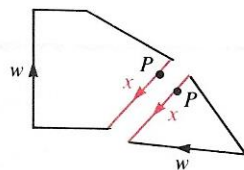
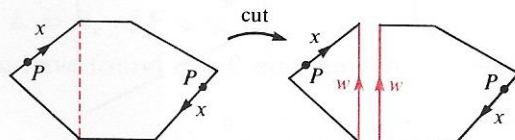
Same definition as Frame 6:

as B_r shrinks to P , K_r shrinks to $f(P) \Leftrightarrow f$ continuous at P .

FRAME 16

P on an (identified) edge of the polygonal representation

polygon representing S



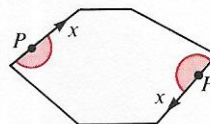
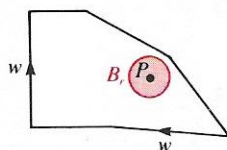
new polygon representing S

FRAME 17

**P on an (identified) edge of the polygonal representation
(no cutting and re-assembling)**

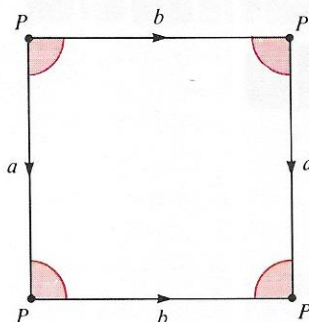
new polygon (as in Frame 15)

original polygon



FRAME 18

P at a vertex of the polygonal representation



FRAME 19

Summary of modifications (to Frame 6) required for special cases

Continuity of $f: X \rightarrow Y$ at P .

- (A) X and Y proper subsets of $\mathbb{R}^2$: instead of $f(B_r)$, consider $f(B_r \cap X)$ as in Frame 14.
- (B) X and Y polygonal representations of two surfaces (still proper subsets of $\mathbb{R}^2$, but with some edges possibly identified):
 - (i) P interior point of X — Frame 15;
 - (ii) P on an (identified) edge — Frame 16 or 17;
 - (iii) P at a vertex — Frame 18;
 - (iv) P on a genuine boundary (non-identified edge) — Frame 14.

UNIT 4

CANONICAL EQUATIONS AND CONNECTED SUMS OF SURFACES

4.1 The Algebra of Lipin Equations

We have seen that the Lipin equation is a polynomial equation in the variables x, y, z and that it is homogeneous of degree n . We now want to study the algebra of Lipin equations.

A Lipin equation is homogeneous of degree n if and only if it is a sum of terms of degree n . This means that if L is a Lipin equation, then L can be written as a sum of terms of degree n . For example, the following equation is a Lipin equation of degree 4:

$$x^4 + y^4 + z^4 + 2x^2y^2 + 2y^2z^2 + 2z^2x^2 = 0$$

This equation is homogeneous of degree 4 because each term has degree 4. We can also write this equation as a sum of terms of degree 4:

can be given as a Lipin equation of degree n if and only if it is a sum of terms of degree n . This means that if L is a Lipin equation, then L can be written as a sum of terms of degree n . For example, the following equation is a Lipin equation of degree 4:

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4.0 Introduction

We have already said that every surface can be represented by edge equations, and that two surfaces are called *combinatorially equivalent* if the edge equations representing the surface are combinatorially equivalent; that is, one can convert a polygonal representation of one surface (given by its edge equations) into a polygonal representation of the other using only the cut and paste operations and their inverses described in *Unit 3*. We shall show that every system of edge equations for a surface can be reduced by these means to one particularly simple equation, called the *canonical equation* of the surface, and we shall prove that two surfaces have the same canonical equation if and only if they are combinatorially equivalent.

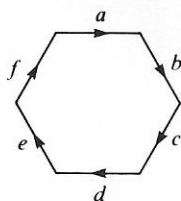
We shall also show how to construct all possible surfaces out of four basic surfaces: the sphere, the torus, the projective plane and the disc. The method of construction was introduced in *Unit 2* and is a way of “adding together” two surfaces to form a new one. This addition operation is called the *connected sum* and provides a direct geometric interpretation of the canonical equation of a surface.

One thing we do not prove in this unit is that two surfaces are homeomorphic if and only if they have the same canonical equation. This is in fact true, but to prove it properly requires ideas of topology plus a good deal of algebra. For that reason we have not included this material in the course. We do explain the underlying geometric ideas, but for their development you will have to study the branch of topology known as “algebraic topology”. Further references are given in Section 4.4.

4.1 The Algebra of Edge Equations

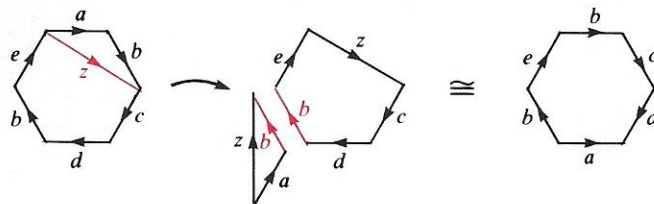
We have met two sets of ways to modify an edge equation of a surface to obtain an equivalent edge equation.

A trivial equivalence corresponds to choosing different routes around the boundary of a polygon, or to renaming edges, or both. Algebraically, this means a succession of transpositions from one side of the equation to the other, or replacing each occurrence of a given letter with another (new) letter, or both. For example, the following surface



can be given an edge equation $abcdef = 1$, or $f^{-1}e^{-1}d^{-1}c^{-1}b^{-1}a^{-1} = 1$ (by transposing all the letters) or $bcdefa = 1$, or $abc = f^{-1}e^{-1}d^{-1}$; and there are many other possibilities. If we rename d as y , this does not change things either, provided that if d (respectively d^{-1}) occurs again we replace the other occurrence by y (respectively y^{-1}). It is the way the edges of the polygon are to be identified that matters, not the choice of letters used to describe them.

A “cut and paste” operation permits certain other rearrangements. For example,



we can cut along z and re-glue along b to obtain an equivalent edge equation

$$\begin{aligned} abcdbe = 1 &\longrightarrow zcdbe = 1; \quad z = ab \\ &\longrightarrow zcdbe = 1; \quad b = a^{-1}z \\ &\longrightarrow zcda^{-1}ze = 1. \end{aligned}$$

Here, b does not appear in the new equation and so we may rename z as b . This does not mean that b refers to the same geometric edge as in the first equation — but remember it is the way the edges are to be matched which interests us, not the particular choice of label. The reason for this particular renaming is that the equation with which we finish up is $bcda^{-1}be = 1$ and this is an *algebraic rearrangement* of the equation with which we started. We have moved from $abcdbe = 1$ to $bcda^{-1}be = 1$. This is an example of the type of algebraic move which we shall study next. In what follows, do not lose sight of our ultimate goal, which is to rearrange an arbitrary edge equation into a standard (canonical) form.

In the first reading passage, **B** shows that we can move whole blocks of edge symbols around an edge equation, according to two fundamental rules, which are stated as Lemmas 1 and 2.

READ B Section 2.3, page 66 to page 68, line – 10.

NOTES (i) *General*

We are moving now towards a systematic algebraic procedure for classifying surfaces. So, although the justification for each move is firmly rooted in geometry, we shall start to perform these algebraic moves according to Lemmas 1 and 2 without appealing to geometry at each stage.

(ii) *Lemma 1*

Notice that the second part of the lemma follows trivially from the first, for

$$\begin{aligned} AxCB^{-1}xD = 1 &\longrightarrow D^{-1}x^{-1}BC^{-1}x^{-1}A^{-1} = 1 \quad (\text{transposing all symbols}) \\ &\longrightarrow D^{-1}x^{-1}C^{-1}x^{-1}B^{-1}A^{-1} = 1 \quad (\text{using the first part of} \\ &\hspace{15em} \text{the lemma to move } B \\ &\hspace{15em} \text{from behind } x^{-1}) \\ &\longrightarrow ABxCxD = 1 \quad (\text{transposing all symbols} \\ &\hspace{15em} \text{again}). \end{aligned}$$

Similarly, the second part of Lemma 2 follows from the first (see SAQ 1 below).

(iii) *Page 68, line 14*

This summary of the lemmas is well worth thinking about. It provides a kind of “pattern recognition” summary which is much easier to remember than the precise statement of the lemmas in terms of equations.

(iv) *Page 68, line – 12*

“Eliminate the letter x ” means rearrange the equation we start with (using transpositions) to the form $x = A$, where A is a block of edge symbols. Then replace the second occurrence of x (or x^{-1}), which is in another equation of the system, by A (or A^{-1}). ▲

SAQ 1 Prove that the second part of Lemma 2 follows from the first part.

SOLUTION TO SAQ 1

$$\begin{aligned} ABxCx^{-1}D = 1 &\longrightarrow Cx^{-1}DABx = 1 \quad (\text{trivial equivalence, cycling symbols}) \\ &\longrightarrow Cx^{-1}BD Ax = 1 \quad (\text{using the first part of Lemma 2} \\ &\hspace{15em} \text{to interchange the blocks } DA \text{ and} \\ &\hspace{15em} B \text{ between } x^{-1} \text{ and } (x^{-1})^{-1} = x) \\ &\longrightarrow Ax Cx^{-1}BD = 1 \quad (\text{by trivial equivalence}). \blacksquare \end{aligned}$$

EXAMPLE 1 For the edge equation $abacdeb^{-1}cfef^{-1}d = 1$:

- (i) use Lemma 1 to move the block ac from before d ;
- (ii) use Lemma 1 to move the block fef^{-1} from before d ;
- (iii) use Lemma 2 to move ac from after b .

Solution

- (i) Lemma 1 applies because d occurs twice in the same sense. Thus we can move the block $B = ac$ from its position in front of the first d to in front of the second, provided that we reverse the sense and write $B^{-1} = c^{-1}a^{-1}$. Thus

$$\begin{array}{ccccccc} A & B & x & C & x & (D = \emptyset) & A & x & C & B^{-1} & x \\ ab & ac & d & eb^{-1}cfef^{-1} & d & = 1 & \longrightarrow & ab & d & eb^{-1}cfef^{-1} & c^{-1}a^{-1} & d & = 1. \end{array}$$

- (ii) This time fef^{-1} appears before the second d and we move it to appear as $fe^{-1}f^{-1}$ before the first d . Thus

$$\begin{array}{ccccccc} A & x & C & B^{-1} & x & (D = \emptyset) & A & B & x & C & x \\ abac & d & eb^{-1}c & fef^{-1} & d & = 1 & \longrightarrow & abac & fe^{-1}f^{-1} & d & eb^{-1}c & d & = 1. \end{array}$$

- (iii) Lemma 2 tells us that we can move the block ac , unchanged, from its position after b to immediately before b^{-1} . Thus

$$\begin{array}{ccccccc} A & x & B & C & x^{-1} & D & A & x & C & B & x^{-1} & D \\ a & b & ac & de & b^{-1} & cfef^{-1}d & = 1 & \longrightarrow & a & b & de & ac & b^{-1} & cfef^{-1}d & = 1. \quad \square \end{array}$$

The next SAQs are a prelude to the second reading passage. In working through them, do not use trivial equivalences unless they are asked for. For example, “move a from before b ” intends the use of Lemma 1 or 2 rather than simply cycling the symbols so that b occurs before a .

SAQ 2 Consider the following edge equation: $cba^{-1}cb^{-1}da = 1$.

Perform the following algebraic moves *consecutively* (i.e. carry out part (ii) on the result of part (i), and so on).

- (i) Move a^{-1} from before c .
- (ii) Move b^{-1} from after c .
- (iii) Move bb from before c .
- (iv) Move a (trivially) from after d .

SAQ 3 Perform the following sequence of moves on the edge equation

$$a^{-1}b^{-1}cadc^{-1}d^{-1}b = 1.$$

- (i) Move $b^{-1}ca$ from in front of d .
- (ii) Move $b^{-1}c$ from after d^{-1} .
- (iii) Move ab (trivially) to before a^{-1} .

**SOLUTION
TO SAQ 2**

$$\begin{array}{ccccccc} x & C & B^{-1} & x & D & (A = \emptyset) & B & x & C & x & D \\ c & b & a^{-1} & c & b^{-1}da & = 1 & \xrightarrow{\text{Lemma 1}} & a & c & b & c & b^{-1}da & = 1 \end{array}$$

$$\begin{array}{ccccccc} A & x & C & x & B^{-1} & D & A & x & B & C & x & D \\ a & c & b & c & b^{-1} & da & = 1 & \xrightarrow{\text{Lemma 1}} & a & c & b & b & c & da & = 1 \end{array}$$

$$\begin{array}{ccccccc} A & x & B^{-1} & x & D & (C = \emptyset) & A & B & x & x & D \\ a & c & bb & c & da & & \xrightarrow{\text{Lemma 1}} & a & b^{-1}b^{-1} & c & c & da & = 1 \end{array}$$

$$\xrightarrow{\text{trivial}} aab^{-1}b^{-1}ccd = 1. \quad \blacksquare$$

**SOLUTION
TO SAQ 3**

$$\begin{array}{cccccc}
 A & B & x & C & x^{-1} & D \\
 a^{-1} & b^{-1}ca & d & c^{-1} & d^{-1} & b = 1
 \end{array}
 \xrightarrow{\text{Lemma 2}}
 \begin{array}{cccccc}
 A & x & C & x^{-1} & B & D \\
 a^{-1} & d & c^{-1} & d^{-1} & b^{-1}ca & b = 1
 \end{array}$$

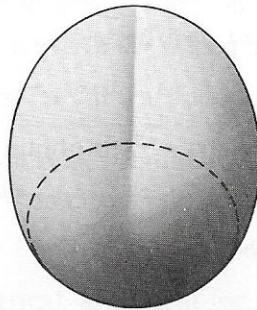
$$\begin{array}{cccccc}
 A & x & C & x^{-1} & B & D \\
 a^{-1} & d & c^{-1} & d^{-1} & b^{-1}c & ab = 1
 \end{array}
 \xrightarrow{\text{Lemma 2}}
 \begin{array}{cccccc}
 A & B & x & C & x^{-1} & D \\
 a^{-1} & b^{-1}c & d & c^{-1} & d^{-1} & ab = 1
 \end{array}$$

$$\xrightarrow{\text{trivial}} aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1. \quad \blacksquare$$

In SAQ 3, notice that no letter occurs twice in the same direction in the original equation. Thus we cannot use Lemma 1; and since Lemma 2 does not reverse the sense of any letters, this state of affairs will not change — Lemma 1 is not applicable.

This suggests that orientability is preserved by combinatorial equivalences, and in Section 4.2 we shall see that this is indeed true.

The final form of the edge equation in SAQ 3 consists of two blocks of symbols like $xyx^{-1}y^{-1}$. You may remember that $xyx^{-1}y^{-1} = 1$ represents a torus, and in SAQ 10 of *Unit 3* we saw that the equation $xyx^{-1}y^{-1} = a$ represents a torus with a disc removed. **B**'s name for a block of symbols of the form $xyx^{-1}y^{-1}$ is a *handle*. You will see why in Section 4.3, but you may remember from TV1 that a torus is a sphere with a handle attached, and **B**'s terminology is based on this. In SAQ 2, there are several blocks of the form cc in the final form of the edge equation. Once again, you will recall that $cc = 1$ represents the projective plane; and the equation $cc = a$ represents a Möbius band. In *Unit 3* we also saw that the projective plane is a disc with a Möbius band attached, and the easiest way to model this construction in $\mathbb{R}^3$ is to attach a *crosscap* to the boundary of a disc.



The crosscap is just a model of the Möbius band in which the boundary circle has been “untwisted” (and which therefore suffers the defect of having a line of self-intersection points). **B** refers to blocks of symbols such as cc as crosscaps. In Section 4.3 we shall explore another geometric interpretation. In the next reading passage, **B** develops the methods we have been using to obtain the canonical edge equation of a surface. The procedure is in four steps. Step 1 shows how all the crosscaps can be assembled at the beginning of the equation. Step 2 shows how the handles are assembled.

READ B from page 68, line -9 to page 70, line 5.

NOTES (i) *Page 68, line -6*

The intention is to bring together at the beginning all the pairs of symbols, such as $\dots s \dots s \dots$ which occur in the same sense in the edge equation. In the given equation $ABsCsD = 1$, the initial block, A , should be thought of as all the pairs $c_1c_1c_2c_2\dots$ which have already been moved to the front. What we have to do is first to bring the next pair ss together, and then adjoin it to the right-hand end of A .

(ii) Page 69, line 12

To understand the format of the equation

$$ABsCtDs^{-1}Et^{-1}F = 1,$$

imagine that A consists of all pairs of symbols $c_1c_1 \dots c_kc_k$ occurring in the same sense (brought together by repeated use of Step 1) and then a number of handles — blocks like $xyx^{-1}y^{-1}$. The object now is to append another handle, $sts^{-1}t^{-1}$, to the right of A . The blocks B, C, D, E and F (some of which may be empty) are other symbols separating the symbols A, s, t, s^{-1}, t^{-1} that we are trying to bring together. (It may happen that some letters x, y occur in the pattern $\dots x \dots y \dots y^{-1} \dots x^{-1} \dots$ where the occurrences of y do not separate those of x . We leave these letters to a later step.) ▲

EXAMPLE 2 Apply Steps 1 and 2 of the reading passage to the single edge equation $ab^{-1}cacdebe^{-1}d = 1$.

Solution

In Step 1 we immediately have a choice, because each of the symbols, a, c, d occurs twice in the same sense. We choose to start with a . (Starting with c or d may lead to a different, but equivalent, final equation, as we shall prove in Section 4.2.) Moving $b^{-1}c$ from behind the first a to behind the second gives

$$aac^{-1}bcdebe^{-1}d = 1.$$

Now moving c^{-1} from before the first b to before the second gives

$$aabcdecbe^{-1}d = 1,$$

and moving the block $cdec$ past b we arrive at

$$aabbce^{-1}d^{-1}c^{-1}e^{-1}d = 1.$$

Then we can move $e^{-1}d^{-1}$ past c^{-1} to obtain

$$aabbce^{-1}c^{-1}dee^{-1}d = 1$$

which is equivalent to $aabbce^{-1}c^{-1}dd = 1$.

EXAMPLE 3 After the first use of Step 1 in Example 2, we found the edge equation

$$aac^{-1}bcdebe^{-1}d = 1.$$

Show that this is equivalent to the edge equation $aaddc^{-1}bceb^{-1}e^{-1} = 1$. Using **B**'s method of assembling handles, show that this can be arranged as

$$aaddc^{-1}bcb^{-1} = 1.$$

Solution

Moving $c^{-1}bc$ from before the first d we obtain

$$aadebe^{-1}c^{-1}b^{-1}cd = 1,$$

and moving the whole block $ebe^{-1}c^{-1}b^{-1}c$ past the second d gives

$$aaddc^{-1}bceb^{-1}e^{-1} = 1.$$

Assembling a handle is a little more complicated, so we write out the argument using **B**'s notation as a check.

$$\begin{array}{cccccccc} A & s & t & s^{-1} & E & t^{-1} & F & (B = C = D = \emptyset) \\ aadd & c^{-1} & b & c & e & b^{-1} & e^{-1} & = 1 \end{array} \longrightarrow \begin{array}{cccccccc} A & s & t & E & s^{-1} & t^{-1} & F \\ aadd & c^{-1} & b & e & c & b^{-1} & e^{-1} & = 1 \end{array}$$

$$\begin{array}{cccccccc} A & s & t & E & s^{-1} & t^{-1} & F \\ aadd & c^{-1} & b & e & c & b^{-1} & e^{-1} & = 1 \end{array} \longrightarrow \begin{array}{cccccccc} A & s & E & t & s^{-1} & t^{-1} & F \\ aadd & c^{-1} & e & b & c & b^{-1} & e^{-1} & = 1 \end{array}$$

$$\begin{array}{cccccccc} A & s & E & t & s^{-1} & t^{-1} & F \\ aadd & c^{-1} & e & b & c & b^{-1} & e^{-1} & = 1 \end{array} \longrightarrow \begin{array}{cccccccc} A & s & t & s^{-1} & t^{-1} & E & F \\ aadd & c^{-1} & b & c & b^{-1} & e & e^{-1} & = 1 \end{array}$$

Cancelling ee^{-1} , we obtain $aaddc^{-1}bcb^{-1} = 1$, as required. $\square$

We deduce from Example 3 that the two equations

$$aabbcc^{-1}c^{-1}dd = 1,$$

$$aaddc^{-1}bcb^{-1} = 1,$$

must be combinatorially equivalent, since both are equivalent to the first equation in Example 2. Now the top equation has four crosscaps, and the lower one has two crosscaps and a handle. This apparent difference is easily explained.

Whenever a block is moved according to Lemma 1, the sense of each symbol in the block is reversed. So after such a movement it could happen that a letter which originally occurred once in each sense now occurs twice in the same sense, creating a crosscap. Indeed, we shall discover in the next reading passage that provided at least one crosscap is present on the edge equation of a surface, then *any* handle present can be converted into two crosscaps; this example is a special case of this rule.

READ B from page 70, line 6 to page 73, line 2.

NOTE Page 70, line – 7

B gives another, more geometrical, argument for showing that three crosscaps are equivalent to a crosscap plus a handle. In the diagram, the shaded strip on the lower left should be labelled III. $\blacktriangle$

We now know how to assemble all crosscaps or all handles at the beginning of the single edge equation for a surface. It remains to work on any other letters which occur at the end of the equation. We cannot find two occurrences of a letter in the same sense, for then it would be possible to assemble a crosscap. Similarly, we cannot find two pairs of letters appearing as $\dots x \dots y \dots x^{-1} \dots y^{-1} \dots$, or we could assemble another handle. So the cases we must deal with are single letters, and pairs of letters such as

$$\dots x \dots y \dots y^{-1} \dots x^{-1} \dots$$

B shows that it is possible to reduce all these possibilities to a sequence of blocks of the form ded^{-1} , which he refers to as *cuffs*.

READ B from page 73, line 3 to page 74, line 10.

NOTES (i) *Page 73, line 16*

The step is to move B , which precedes d , so that it follows d^{-1} .

(ii) *Page 73, line -12*

The rather complicated argument which follows is solely directed to proving the equivalence of $Ad_1e_1d_1^{-1} \dots d_re_r d_r^{-1}e = 1$

$$\text{with } Ad_1e_1d_1^{-1} \dots d_re_r d_r^{-1} ded^{-1} = 1.$$

You might like to refer back to SAQ 10 of *Unit 3* where this was carried out in a particular example. The algebraic argument corresponds to the geometrically obvious process of adding a cuff to the boundary curve e .

(iii) *Page 73, line -9*

"By two transpositions": it is easier to think of this as simply commencing the description of the polygon corresponding to the edge equation with d instead of with the first symbol (edge) in A .

(iv) *Page 73, line -1*

B has made a mistake here in omitting a necessary step between his second and third. The correct sequence is

$$\begin{aligned} da_1b_1a_1^{-1}b_1^{-1} &\xrightarrow{\text{Lemma 2}} a_1b_1a_1^{-1}db_1^{-1} \\ &\xrightarrow{\text{Lemma 2}} a_1b_1da_1^{-1}b_1^{-1} \\ &\xrightarrow{\text{Lemma 2}} a_1db_1a_1^{-1}b_1^{-1} \\ &\xrightarrow{\text{Lemma 2}} a_1b_1a_1^{-1}b_1^{-1}d. \quad \blacktriangle \end{aligned}$$

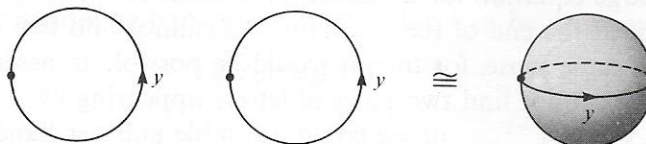
The final reading passage in this section summarizes the four steps and describes what has been discovered geometrically.

READ B page 74, line 11 to the end of Section 2.3.

NOTES (i) *Page 74, last paragraph*

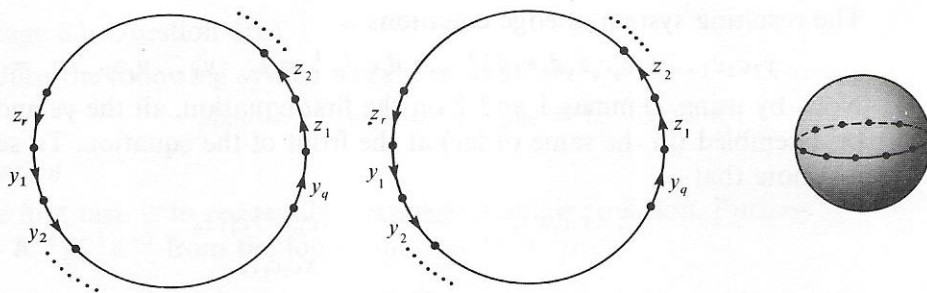
An alternative way to *see* the geometrical significance of the canonical forms is to start with a sphere, cut out a number of (open) discs and glue either crosscaps and cuffs, or handles and cuffs, onto the boundaries of the cuts. We shall show that this procedure yields surfaces whose edge equations are precisely the canonical forms.

Start with two discs ($y = 1; y = -1$) glued along their common boundary curve y to form a sphere. (You can think of the discs as the northern and southern hemispheres, meeting at the equator y .)

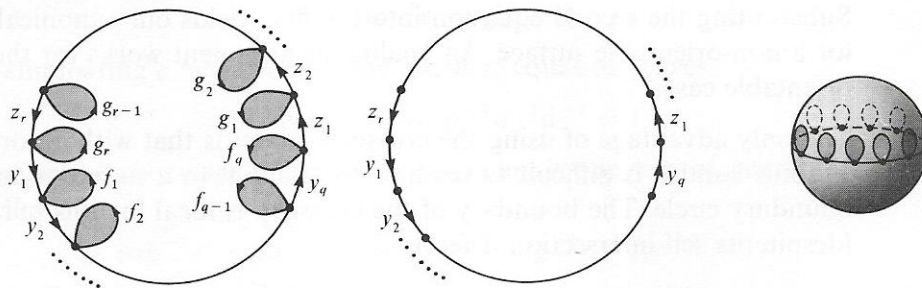


We are now going to cut $q + r$ open discs out of this sphere and glue q crosscaps and r cuffs to the boundary circles thus obtained. To this end, introduce $q + r - 1$ new vertices on y (giving $q + r$ altogether) and label the new edges to give the system

$$y_1y_2 \dots y_qz_1 \dots z_r = 1; \quad y_1 \dots y_qz_1 \dots z_r = 1.$$



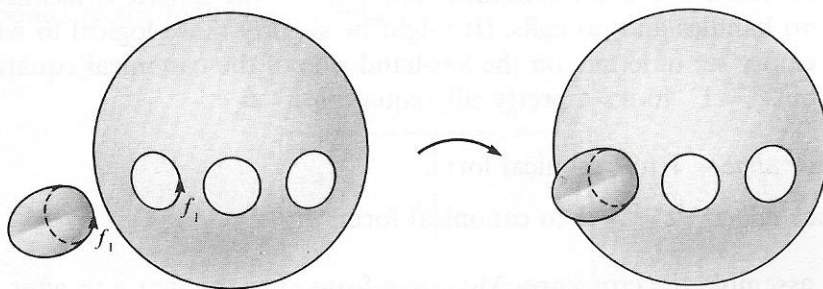
Now cut the first disc along new edges f_i, g_j ($1 \leq i \leq q, 1 \leq j \leq r$), and remove open discs.



The new system of edge equations is

$$y_1 f_1 y_2 f_2 \dots y_q f_q z_1 g_1 \dots z_r g_r = 1; \quad y_1 y_2 \dots y_q z_1 \dots z_r = 1.$$

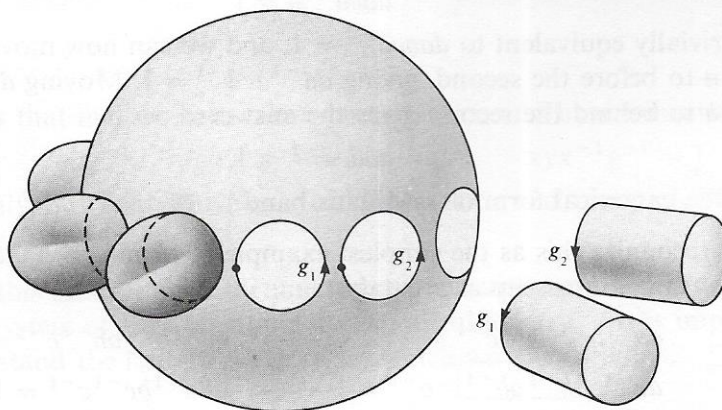
This has $q + r$ boundary curves, namely $f_1, \dots, f_q, g_1, \dots, g_r$. Now take a crosscap, $f_1 = c_1 c_1$, with boundary curve f_1 , and glue this to the boundary curve f_1 on the punctured sphere.



Repeating this for each f_i gives the following system of edge equations:

$$y_1 c_1 c_1 y_2 \dots y_q c_q c_q z_1 g_1 \dots z_r g_r = 1; \quad y_1 \dots y_q z_1 \dots z_r = 1.$$

Next, glue a cuff $g_i = d_i e_i d_i^{-1}$ to each remaining boundary curve g_i .



The resulting system of edge equations is

$$y_1 c_1 c_1 \dots y_q c_q c_q z_1 d_1 e_1 d_1^{-1} \dots z_r d_r e_r d_r^{-1} = 1; \quad y_1 \dots y_q z_1 \dots z_r = 1.$$

Now, by using Lemmas 1 and 2 on the first equation, all the y s and z s may be assembled (in the same order) at the front of the equation. To see this, just note that

$$\dots c_i c_i x \dots \xrightarrow{\text{Lemma 1}} \dots c_i x^{-1} c_i \dots$$

$$\xrightarrow{\text{Lemma 1}} \dots x c_i c_i \dots,$$

and

$$\dots d_j e_j d_j^{-1} x \xrightarrow{\text{Lemma 2}} \dots x d_j e_j d_j^{-1}$$

Thus the above system is equivalent to

$$y_1 \dots y_q z_1 \dots z_r c_1 c_1 \dots c_q c_q d_1 e_1 d_1^{-1} \dots d_r e_r d_r^{-1} = 1; \quad y_1 \dots y_q z_1 \dots z_r = 1.$$

Substituting the second equation into the first yields our canonical form for a non-orientable surface. An analogous argument works for the orientable case.

The only advantage of using the crosscap model is that with an ordinary Möbius band it is difficult to see how to attach it to a surface along its boundary circle. The boundary of the crosscap is ideal for this purpose (despite its self-intersection defect).

(ii) Page 74, line - 2

It is worth remembering that one edge equation for the sphere is $ww^{-1} = 1$ (just make a single cut in a sphere from the north pole to the south pole), and by a trivial equivalence (corresponding to mending a cut) we can cancel w and w^{-1} . But what do we write when the edge equation is $ww^{-1} = 1$ to begin with? **B** suggests $1 = 1$ for the solution to this puzzle, so that $1 = 1$ is the canonical equation for the sphere. It has no crosscaps, no handles and no cuffs. (It might be slightly more logical to write the empty set of letters on the left-hand side of the canonical equation, but $=1$ looks a pretty silly equation.) ▲

SAQ 4 Reduce $abac = 1$ to canonical form.

SAQ 5 Reduce $aabcb^{-1}c^{-1} = 1$ to canonical form.

SOLUTION TO SAQ 4 First assemble the crosscaps. Moving b from after the first a to after the second a gives

$$aab^{-1}c = 1.$$

Replacing $b^{-1}c$ by a new edge e gives $aae = 1$, and adding a pair $d^{-1}d$ we obtain

$$aaed^{-1}d = 1.$$

This is trivially equivalent to $daaed^{-1} = 1$, and we can now move d from before the first a to before the second, giving $ad^{-1}aed^{-1} = 1$. Moving d^{-1} from behind the first a to behind the second gives the answer

$$aaded^{-1} = 1.$$

(This is the canonical form of a Möbius band.) ■

SOLUTION TO SAQ 5 You will recognize this as the simplest example of turning a handle into two crosscaps using a crosscap as a catalyst.

$$\begin{array}{lcl} aa \boxed{bc} b^{-1}c^{-1} = 1 & \longrightarrow & ac^{-1}b^{-1}ab^{-1}c^{-1} = 1 \\ ac^{-1} \boxed{b^{-1}ab^{-1}} c^{-1} = 1 & \longrightarrow & aba^{-1}bc^{-1}c^{-1} = 1 \\ ab \boxed{a^{-1}} bc^{-1}c^{-1} = 1 & \longrightarrow & aabbc^{-1}c^{-1} = 1, \end{array}$$

and renaming c^{-1} as c gives the result. ■

We give one final example at length. There are many problems based on the material of this section in the Exercise Booklet, and you are advised to do some to gain practice.

EXAMPLE 4 B page 83, Question 1(b).

Reduce the following system to canonical form:

$$abcd = 1; \quad eafc = 1; \quad gfhe = 1; \quad hdgb = 1.$$

Solution

The first task is to reduce the system to a single equation. Putting $h = b^{-1}g^{-1}d^{-1}$ from the fourth into the third gives

$$gfb^{-1}g^{-1}d^{-1}e = 1,$$

and from this we see that $f = g^{-1}e^{-1}dgb$. Putting this in the second equation gives

$$eag^{-1}e^{-1}dgb = 1,$$

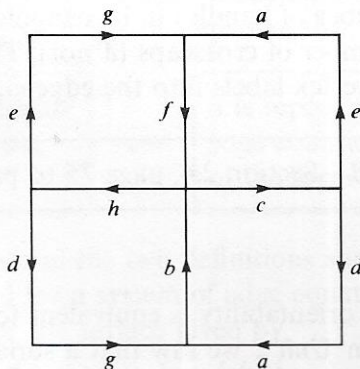
and eliminating c between this and the first equation leaves

$$eag^{-1}e^{-1}dgb b^{-1}a^{-1}d^{-1} = 1.$$

There are no crosscaps present. The following is one way of assembling the handles.

$$\begin{aligned} eag^{-1}e^{-1}dga^{-1}d^{-1} = 1 &\longrightarrow eg^{-1}e^{-1}dgd^{-1} = 1 \\ eg^{-1}e^{-1}dgd^{-1} = 1 &\longrightarrow d^{-1}eg^{-1}e^{-1}dg = 1 \\ d^{-1}eg^{-1}e^{-1}dg = 1 &\longrightarrow d^{-1}deg^{-1}e^{-1}g = 1 \\ &\longrightarrow eg^{-1}e^{-1}g = 1 \quad \text{— a torus!} \end{aligned}$$

In this particular example there happens to be a very easy way to check that we do, indeed, have a torus. Start with the four squares represented by the original edge equations. Then the reduction to a single equation which we chose corresponds to the following glueing.



This shows that had we been clever enough we could have converted

$$eag^{-1}e^{-1}dga^{-1}d^{-1} = 1 \quad \text{into} \quad xyx^{-1}y^{-1} = 1$$

by using only the trivial equivalence of transposition and operation 3 to replace $e^{-1}d$ by x and ga^{-1} by y !

However, this kind of good fortune cannot be relied upon to reduce an arbitrary system of edge equations to canonical form, and it is important that you understand the four-step algorithm which we have studied. $\square$

In Section 4.2 we shall find a more efficient way of *calculating* the canonical form of a given system of edge equations. The real role of canonical forms is to prove the classification theorem (and to provide an agreed notation for the standard list of surfaces). You should not feel constrained to use canonical forms when working problems. (For example, $y = 1$ is a simpler version of a disc than is $ded^{-1} = 1$, and $y = cc$ is an easier Möbius band than is $ccded^{-1} = 1$.)

4.2 Combinatorial Invariants of a Surface

We now know that any surface can be represented by a system of edge equations, and that this system can be transformed by a sequence of combinatorial operations into canonical form. There remains the following loose end. In several examples, we have seen that there is choice as to which particular sequence of moves we use to reduce the given equation to canonical form. Unfortunately, it is not at all clear that two different reduction procedures might not produce different canonical forms. The next reading passage disposes of this possibility, and as a by-product also presents a much more efficient algorithm for finding the canonical form of a given system of edge equations.

Recall that the classification theorem we are trying to prove says that a surface is determined by three things:

Euler characteristic;
number of boundary curves;
orientability.

What we do next is to show that:

- (i) each of these can be found *directly* from the edge equations of the surface;
- (ii) when we perform *any* of the combinatorial equivalences on a system of edge equations, the above three properties *do not change*; and
- (iii) different canonical forms differ in at least one of these three topological invariants.

It follows that no finite sequence of combinatorial equivalences can change the three properties either, and so we cannot get two different canonical forms starting from the same system of edge equations.

B starts with orientability and then goes on to consider *not* the Euler characteristic, but a related number, called the *genus* of the surface. This is defined to be the number of handles in its canonical equation (if the surface is orientable), or the number of crosscaps (if not). The easiest way to compute these is to introduce vertex labels into the edge equations.

READ B Section 2.4, page 75 to page 77, line – 13.

NOTES (i) *Page 75, line 19*

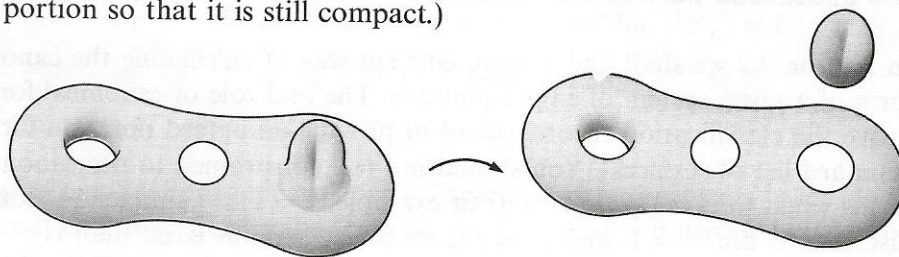
The definition of orientability is equivalent to our definition in *Unit 2*. Remember that in *Unit 2* we saw that a surface is *non-orientable* in our sense if, and only if, it contains a Möbius band. We also saw (SAQ 15 of *Unit 2*) that if a surface is represented by an edge equation of the form

$$\dots a \dots a \dots = 1,$$

then it certainly does contain a Möbius band. It is not hard to see that the converse of this is also true, namely that if a surface S contains a Möbius band then it is represented by an edge equation of the form

$$\dots a \dots a \dots = 1.$$

To see this, cut the Möbius band out of S by cutting along its boundary circle. (When we do this, we take care to “fill in” the boundary of the remaining portion so that it is still compact.)



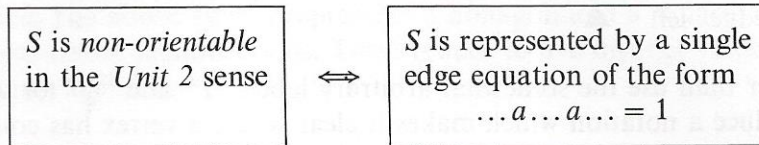
The remaining portion can then be further cut up into topological polygons, and in this way we arrive at a system of edge equations

$$A_1 = 1; \quad A_2 = 1; \quad \dots \quad A_r = 1.$$

One of these is the equation of the Möbius band which we cut first, and thus contains some letter twice in the same sense. Reducing the system to a single edge equation cannot alter this situation, and so the end product is indeed a single edge equation of the form

$$\dots a \dots a \dots = 1.$$

Thus



B's definition deals with a *system* of edge equations. However, since surfaces are, by definition, connected, any such system must be reducible to a single edge equation. Now, assume that S is *orientable* in **B**'s sense, and is thus represented by a system of edge equations of the form

$$A_1 = 1; \quad A_2 = 1; \quad \dots \quad A_r = 1,$$

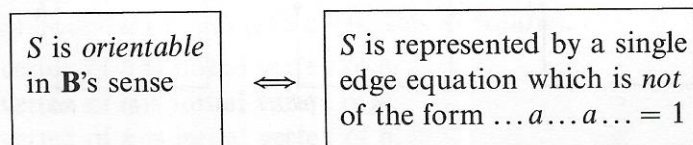
in which no letter appears twice in the same sense. Suppose that A_i and A_j have a letter in common. Then (cycling the letters if necessary) we can write these two equations as

$$xB_i = 1, \quad x^{-1}B_j = 1,$$

where no letter occurs twice in the same sense. Amalgamating these two equations gives the single equation

$$B_j B_i = 1,$$

and this has the same property. Any reduction of the system to a single equation consists of a sequence of steps like this, and so



The logical equivalence of the two definitions should now be apparent. **B**'s definition can be used for a system of edge equations, but care is essential. For example, consider the surface given by

$$abc = 1; \quad acdf = 1; \quad deb^{-1} = 1; \quad gef^{-1} = 1.$$

This is *orientable*, since inverting the second and fourth equations leads to

$$abc = 1; \quad f^{-1}d^{-1}c^{-1}a^{-1} = 1; \quad deb^{-1} = 1; \quad fe^{-1}g^{-1} = 1,$$

and this satisfies **B**'s criterion. The safest practice is always to reduce a system to a single edge equation and examine this for letters occurring twice in the same sense. Eliminating b , c and f from the above system gives rise to the equation

$$ae^{-1}d^{-1}a^{-1}dge = 1,$$

which is clearly orientable (according to either definition).

(ii) Page 77, line -16

"Boundary edges may be arranged into cyclic sets" Suppose e_1 is a boundary edge (i.e. it occurs only once in the edge equations). Either e_1 is itself a complete boundary curve, or its terminal vertex is the initial vertex of just one other boundary edge, e_2 . This is geometrically obvious, but we have to justify it combinatorially. **B** points out in the first sentence of this paragraph that a vertex either does not occur as the endpoint of a

boundary edge or it does so twice (possibly as both endpoints of a single boundary edge). (See **B** page 76, line -9.) The edge e_2 has as terminal vertex the initial vertex of just one boundary edge, e_3 . It may be that $e_3 = e_1$, in which case e_1e_2 is a complete boundary curve. (Why is $e_3 = e_2$ an impossibility?) Otherwise, the terminal vertex of e_3 is the initial vertex of another boundary edge, e_4 , and so the argument continues. The only way this process ends (as it must, since there are only a finite number of edges) is by eventually returning to the initial vertex of e_1 . We then have a complete boundary curve $e_1e_2 \dots e_n$. This forms one of the "cyclic sets" to which **B** refers. We now repeat the procedure starting with any unclaimed boundary edge, and continue in this way until there are no more boundary edges left. ▲

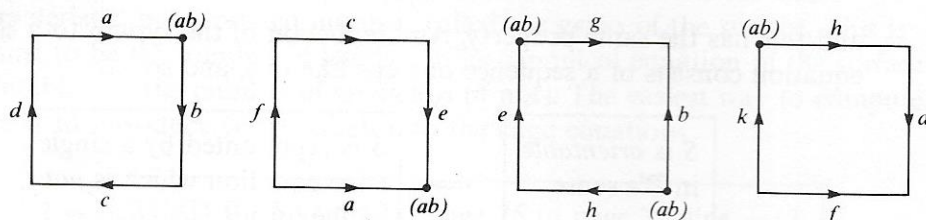
Rather than use the somewhat arbitrary labels " P " and " Q " for vertices, we can introduce a notation which makes it clear where a vertex has come from. If the edges a, b occur consecutively in an edge equation, we denote their common vertex by (ab) (i.e. the terminal vertex of a and initial vertex of b). The same vertex will occur elsewhere either as the initial vertex of a^{-1} or as the terminal vertex of b^{-1} , provided that one of these is present in an edge equation. Notice also that an equation of the form $aXz = 1$ can also be written as $Xza = 1$, and so we can reasonably write (za) for the terminal vertex of z and initial vertex of a .

EXAMPLE 5 To illustrate the process of inserting vertex symbols and forming sequences of vertices like those **B** describes, consider the surface

$$abcd = 1; \quad cea^{-1}f = 1; \quad gb^{-1}he = 1; \quad hdf^{-1}k = 1.$$

Show that this has three vertices.

Solution



Starting at (ab) , we look for other occurrences of a and b . (ab) is the terminal vertex of a and hence the initial vertex of a^{-1} . Hence $(ea^{-1}) = (ab)$. Similarly, $(eg) = (ab)$, and since g appears only once, the sequence terminates here. Going in the other direction from (ab) , which is the initial vertex of b (and therefore the terminal vertex of b^{-1}), we find

$$(ab) = (b^{-1}h) = (kh).$$

Since k appears only once, this is the full sequence.

There are several possible vertices unaccounted for, so we repeat the procedure with a new one, such as (bc) . Starting with the second occurrence of b we obtain $(bc) = (gb^{-1})$, and the sequence stops here because g appears only once. Going in the other direction gives

$$(bc) = (fc) = (df^{-1}) = (da) = (a^{-1}f) = (f^{-1}k),$$

and here the sequence terminates.

Looking through the list so far it is evident that (cd) does not appear, so we repeat the procedure.

$$(cd) = (ce) = (he) = (hd) = (cd),$$

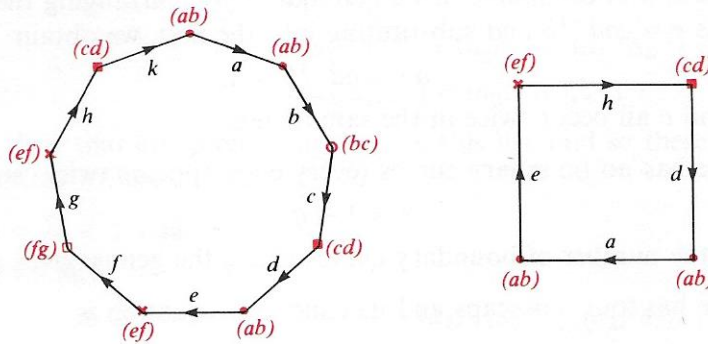
and here the sequence repeats itself.

The original system of edge equations represents four squares, with a total of 16 vertices before identifications. All 16 possibilities appear in our list, which is therefore complete. There are only three distinct vertices when the surface is assembled. Notice also that the boundary edges form a cyclic set of length 2: the terminal vertex of g (i.e. (gb^{-1})) is the initial vertex of k (i.e. $(f^{-1}k)$), and the terminal vertex of k ($((kh))$) is also the initial vertex of g ($((eg))$).

SAQ 6 Determine the vertices and boundary curves of the surface given by

$$abcdefghk = 1; \quad hdae = 1.$$

SOLUTION TO SAQ 6 There are five boundary letters, b, c, f, g and k , since the other four letters each appear twice. The above system represents a nonagon and a square, with a total of 13 vertices before identifications. These reduce to five distinct vertices after identifications, as follows.



$(ab) = (ae) = (de) = (da) = (ka)$	(involving boundary edges b, k)
(bc)	(involving boundary edges b, c)
$(cd) = (hd) = (hk)$	(involving boundary edges c, k)
$(ef) = (eh) = (gh)$	(involving boundary edges f, g)
(fg)	(involving boundary edges f, g)

We form the five boundary edges into cyclic sets as follows.

Terminal vertex of b is initial vertex of c .
 Terminal vertex of c is initial vertex of k .
 Terminal vertex of k is initial vertex of b .
 Terminal vertex of f is initial vertex of g .
 Terminal vertex of g is initial vertex of f .

Thus the cyclic sets are bck and fg , and these represent the two boundary curves of the surface. ■

READ B page 77, line - 12 to the end of Section 2.4.

NOTE Page 79, lines 3–5

This verification is straightforward. For example, if the first equation has two crosscaps and two cuffs it is equivalent to

$$aabb d_1 e_1 d_1^{-1} d_2 e_2 d_2^{-1} = 1.$$

Starting with $P = (d_2^{-1}a)$ as the initial vertex of a , we find that $P = (aa)$. So P is also the terminal vertex of a , and hence $P = (ab)$. This continues until we reach $P = (bd_1)$, the initial vertex of d_1 (and therefore the terminal vertex of d_1^{-1}). Thus $P = (d_1^{-1}d_2)$, and the sequence then repeats itself with $P = (d_2^{-1}a)$. We are then left with one sequence of identified vertices for each cuff:

$$Q_1 = (d_1 e_1) = (e_1 d_1^{-1}),$$

$$Q_2 = (d_2 e_2) = (e_2 d_2^{-1}).$$

A similar type of calculation applies for the orientable canonical form (see SAQ 7). ▲

EXAMPLE 6 B pages 83–4, Question 1(a).

Classify the surface

$$abcde = 1; \quad ad^{-1}be^{-1}c = 1.$$

*Solution*We first examine the vertices. Starting with (ab) we have

$$\begin{aligned} (ab) &= (ad^{-1}) = (de) = (e^{-1}c) = (bc) = (be^{-1}) \\ &= (ea) = (ca) = (cd) = (d^{-1}b) = (ab) \end{aligned}$$

and there is only one vertex.

Thus the surface has two faces (polygons), five edges (one for each letter, not counting repetitions) and one vertex. Its Euler characteristic is therefore

$$\chi = 2 - 5 + 1 = -2.$$

The surface is non-orientable: if we eliminate e by rearranging the second equation as $e = cad^{-1}b$ and substituting into the first, we obtain

$$abcdcad^{-1}b = 1,$$

and a , b and c all occur twice in the same sense.

The surface has no boundary curves (every edge appears twice) so the formula

$$\chi = 2 - q - r,$$

where r is the number of boundary curves and q the genus, gives $q = 4$.

The surface has four crosscaps and its canonical equation is

$$c_1c_1c_2c_2c_3c_3c_4c_4 = 1.$$

SAQ 7 Carry out the vertex-labelling procedure for the surface

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}d_1e_1d_1^{-1}d_2e_2d_2^{-1} = 1.$$

SAQ 8 B pages 83–4, Questions 1(b) and 1(c).**SAQ 9** B page 84, Question 2.**SAQ 10** B page 84, Question 3.**SOLUTION TO SAQ 7** Starting with $P = (d_2^{-1}a_1)$, P is the initial vertex of a_1 and hence the terminal vertex of a_1^{-1} . Thus

$$P = (a_1^{-1}b_1^{-1}) = (b_1a_1^{-1}) = (a_1b_1) = (b_1^{-1}a_2), \quad \text{and so on.}$$

We move through the handles, arriving finally at

$$P = (b_2^{-1}d_1) = (d_1^{-1}d_2) = (d_2^{-1}a_1).$$

The remaining vertices split into sequences, one for each cuff, as in the non-orientable case. ■

SOLUTION TO SAQ 8 (b) As before, we start by examining the vertices. Starting with (ab) we have

$$(ab) = (ae) = (de) = (df) = (f^{-1}a) = (ea) = (ef^{-1}) = (fh),$$

and the sequence stops here since h is a boundary edge. Continuing the sequence in the other direction we obtain $(ab) = (hb)$ and the sequence terminates. We have accounted for 9 of the 12 vertices before identification. (bc) is one of the missing ones, and $(bc) = (bd) = (cd)$ is the remaining sequence. There are thus two vertices after identifications.

The Euler characteristic is easily found: there are three faces (polygons) and 7 edges (one for each letter). Thus $\chi = 3 - 7 + 2 = -2$.The surface is orientable, because if we invert the first equation to $e^{-1}d^{-1}c^{-1}b^{-1}a^{-1} = 1$, then in the resulting system no letter appears twice in the same sense.

There are two boundary curves: since the initial vertex of h is also the terminal vertex of h , h is a complete boundary curve. Likewise c .

Now $\chi = 2 - 2p - r$ gives $p = 1$. The surface is a torus with two (open) discs removed, and has canonical equation

$$aba^{-1}b^{-1}d_1e_1d_1^{-1}d_2e_2d_2^{-1} = 1.$$

(c) $a_1a_2a_3 \dots a_na_1^{-1}a_2^{-1}a_3^{-1} \dots a_n^{-1} = 1.$

The surface is orientable and has no boundary curves.

Case 1: $n = 2k$, even.

Following vertices we have

$$(a_1a_2) = (a_2^{-1}a_3^{-1}) = (a_3a_4) = (a_4^{-1}a_5^{-1}) = \dots = (a_{2r-1}a_{2r}) = (a_{2r}^{-1}a_{2r+1}^{-1}).$$

When $r = k$, this becomes

$$\begin{aligned} (a_{n-1}a_n) &= (a_n^{-1}a_1) \\ &= (a_1^{-1}a_2^{-1}) = (a_2a_3) = (a_3^{-1}a_4^{-1}) = \dots \\ &= (a_{2r}a_{2r+1}) = (a_{2r+1}^{-1}a_{2r+2}^{-1}). \end{aligned}$$

It is now clear that every vertex appears in this list, and so there is only one distinct vertex in Case 1.

Case 2: $n = 2k + 1$, odd.

This time we have

$$(a_1a_2) = (a_2^{-1}a_3^{-1}) = \dots = (a_{2r-1}a_{2r}) = (a_{2r}^{-1}a_{2r+1}^{-1}).$$

When $r = k$ this gives

$$(a_{n-1}^{-1}a_n^{-1}) = (a_na_1^{-1}) = (a_1a_2)$$

and the sequence repeats itself without accounting for (a_2a_3) .

Chasing (a_2a_3) around gives all other pairs $(a_{2r}a_{2r+1})$ and $(a_{2r-1}^{-1}a_{2r}^{-1})$.

Thus Case 2 yields two distinct vertices.

In either case there are n edges and one face so the Euler characteristic is

$$\text{Case 1: } 1 - n + 1 = 2 - n = 2(1 - k),$$

$$\text{Case 2: } 1 - n + 2 = 3 - n = 2(1 - k).$$

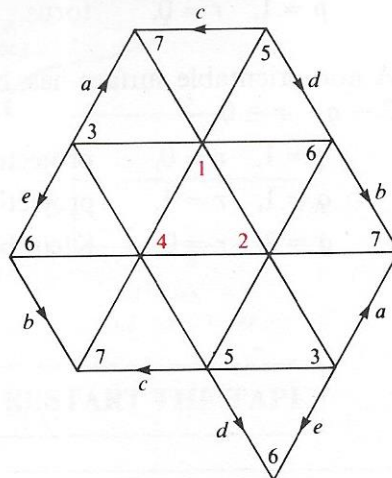
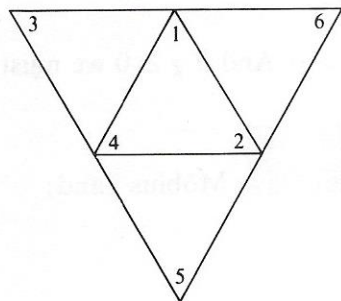
So the surface is a sphere with $\left\lfloor \frac{n}{2} \right\rfloor$ handles. ■

SOLUTION TO SAQ 9

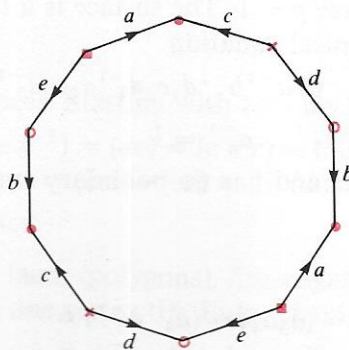
Probably the most systematic way to do this problem is to start with the first triangle, $\triangle 124$, and sketch the subsequent triangles, matching edges wherever possible. After this we shall be left with a system of edge equations and the vertices will be ready labelled.

Starting with $\triangle 124$, we can glue on triangles 134, 245 and 126.

Then we can add 235, 346, 457, 561, 672, 713, 356, 467, 571 and 723 to obtain the following polygon.

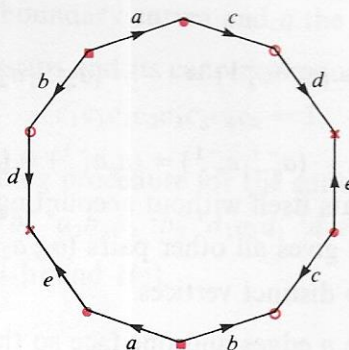


Now we label edges arbitrarily, but taking care to make the correct identifications, and arrive at the following representation.



An edge equation is $ab^{-1}d^{-1}ca^{-1}ebc^{-1}de^{-1} = 1$. This shows that the surface is orientable and has no boundary. We have one face, five edges and four vertices, and the Euler characteristic is therefore $\chi = 1 - 5 + 4 = 0$; the genus is $\frac{1}{2}(2 - \chi) = 1$. The surface is a torus!

We apply exactly the same procedure to Δ_2 , but this time we shall omit some of the working. A possible polygon is



and an edge equation for this is $aed^{-1}b^{-1}acde^{-1}cb^{-1} = 1$. Thus we have one face, five edges and four vertices, giving Euler characteristic zero. The surface is non-orientable and without boundary. Hence the genus is two and the surface is a Klein bottle! ■

SOLUTION TO SAQ 10

An orientable surface with $\chi \geq 0$ has $2 - 2p - r \geq 0$. Since p and r are both non-negative integers, there are obviously very few possibilities, and so we simply list them one by one.

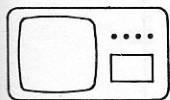
- | | |
|-----------------------|-----------|
| $p = 0, \quad r = 0,$ | sphere; |
| $p = 0, \quad r = 1,$ | disc; |
| $p = 0, \quad r = 2,$ | cylinder; |
| $p = 1, \quad r = 0,$ | torus. |

A non-orientable surface has, by definition, $q > 0$. And if $\chi \geq 0$ we must have $2 - q - r \geq 0$.

- | | |
|-----------------------|---|
| $q = 1, \quad r = 0,$ | projective plane; |
| $q = 1, \quad r = 1,$ | projective plane with a hole = Möbius band; |
| $q = 2, \quad r = 0,$ | Klein bottle. ■ |

4.3 Connected Sums of Surfaces

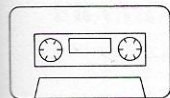
We have seen that every surface is combinatorially equivalent to one whose edge equation is one of the standard list of possible canonical equations. In this section, which includes TV and TV-related Audio, we give a geometrical interpretation of each canonical equation. That is, we see what every possible surface actually *looks* like.



Before watching TV4, you will find it useful to work through the tape section below as far as Frame 4. This takes you through some problems which link the material of previous sections with the programme.

ALLOW AT LEAST 15 MINUTES FOR THIS.

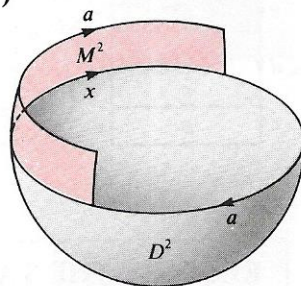
Pre-TV Work



Familiarize yourself with Frame 1, then listen to Track 2, Side 2 of the TV-related Audio-cassette A1, up to the point when you are told that watching TV4 should be your next activity.

FRAME 1

TV3 model of $\mathbb{RP}^2$: a Möbius band M^2 glued to a disc D^2 along their boundaries (i.e. $M^2 \cup_{\partial} D^2$)

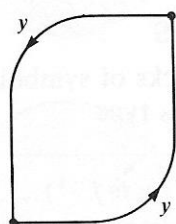


∂ = circle made up of x followed by a

START THE TAPE

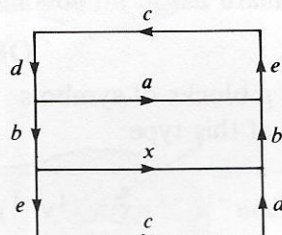
FRAME 2

$\mathbb{RP}^2 \cong M^2 \cup_{\partial} D^2$ using polygons and edge equations



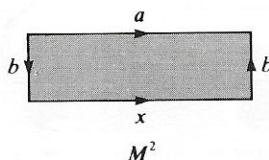
$\mathbb{RP}^2: yy = 1$

$y \rightarrow cdbe$
mark a and x

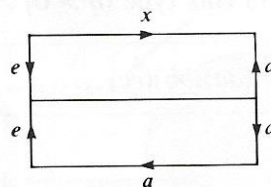


$\mathbb{RP}^2: (cdbe)(cdbe) = 1$

cut along a, x
and glue along c



M^2



$xd^{-1}daee^{-1} = 1$
i.e. $xa = 1 : D^2$

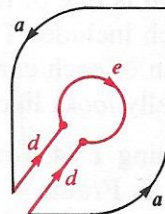
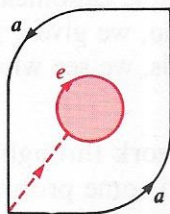
Compare this with Frame 1, then RESTART THE TAPE

FRAME 3

Removing any $\hat{D}^2$ from $\mathbb{RP}^2$ leaves M^2

$\mathbb{RP}^2$ with open disc removed

Allowable representation

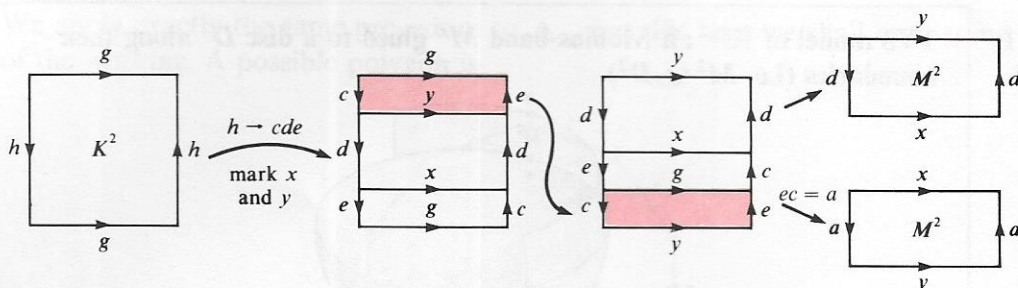


$aaded^{-1} : M^2$

NOTE For reasons of clarity, $\hat{D}^2$ appears in the programme simply as D^2 .

FRAME 4

Klein bottle $K^2 \cong M^2 \cup_{\partial} M^2$: two Möbius bands, glued along their boundaries



RESTART THE TAPE

TV Overview

After watching TV4, read Frame 5 before restarting the tape.

FRAME 5

Standard list of all possible canonical equations (proved in Section 4.2)

ORIENTABLE SURFACES

p blocks of symbols
of this type

r blocks of symbols
of this type

$$(aba^{-1}b^{-1})(xyx^{-1}y^{-1})\dots\dots\dots(ded^{-1})(fgf^{-1})\dots\dots\dots = 1$$

NON-ORIENTABLE SURFACES

q blocks of symbols
of this type ($q > 0$)

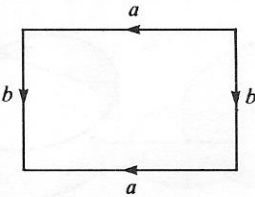
r blocks of symbols
of this type

$$(aa)(bb)(cc)\dots\dots\dots(ded^{-1})(fgf^{-1})\dots\dots\dots = 1$$

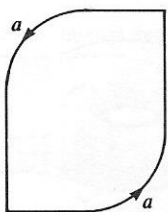
RESTART THE TAPE

FRAME 6

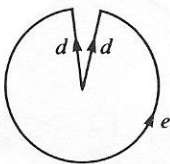
Blocks in canonical equations correspond to three surfaces



$T^2: aba^{-1}b^{-1} = 1$



$RP^2: aa = 1$

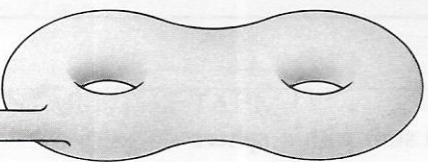
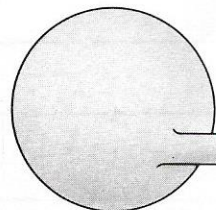
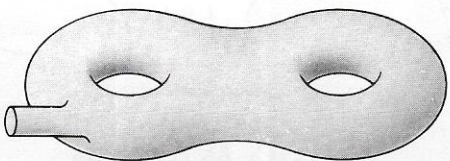
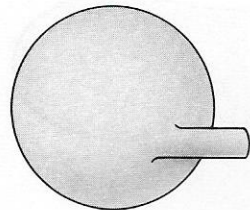
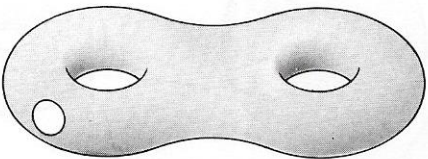
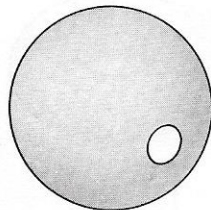
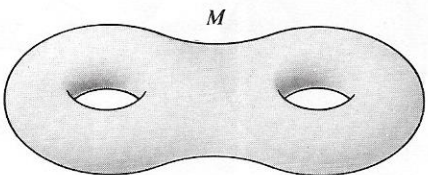
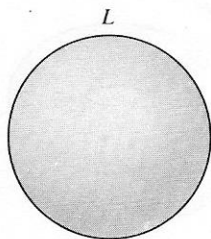


$D^2: ded^{-1} = 1$

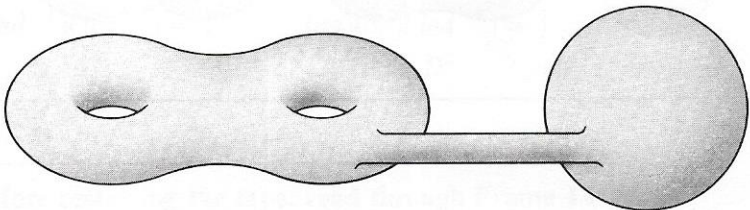
THIS SUGGESTS THAT WE CAN CONSTRUCT ALL SURFACES FROM THESE THREE.

FRAME 7

Method of construction: CONNECTED SUM



The CONNECTED SUM of L and M , $L \# M$

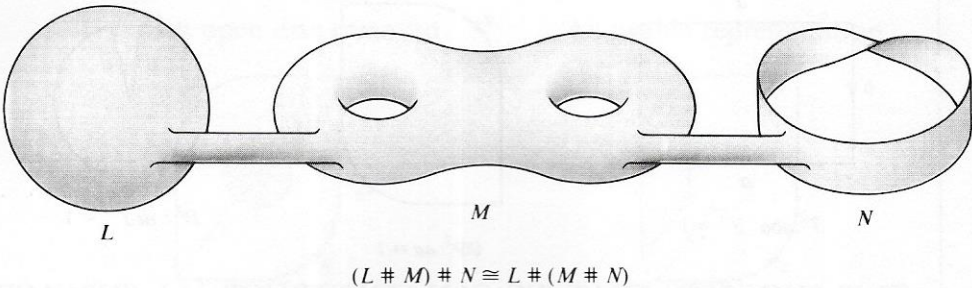


Commutative : $L \# M$ is homeomorphic to $M \# L$

$M \# L$

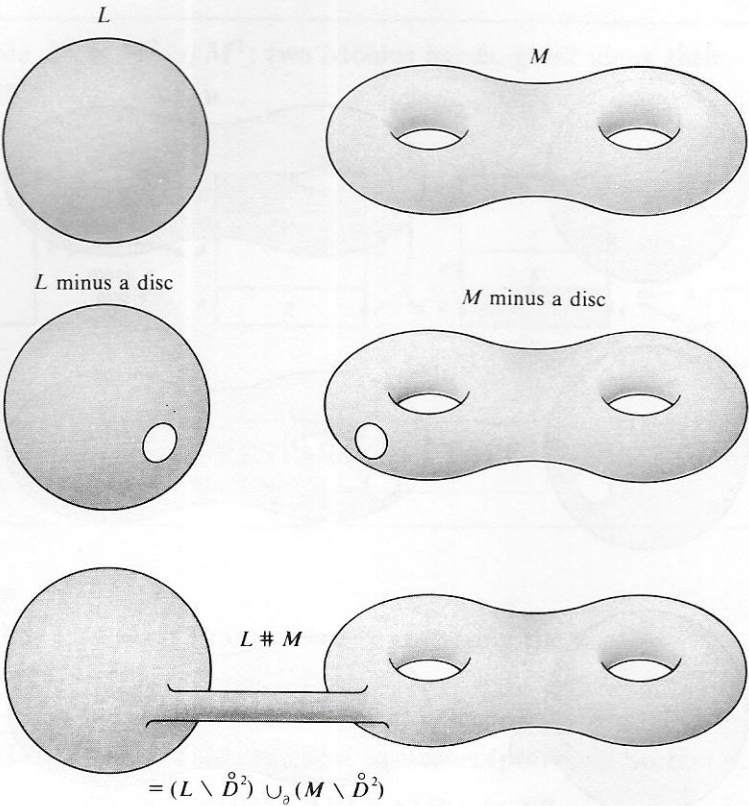
FRAME 8

Connected sum is also associative



FRAME 9

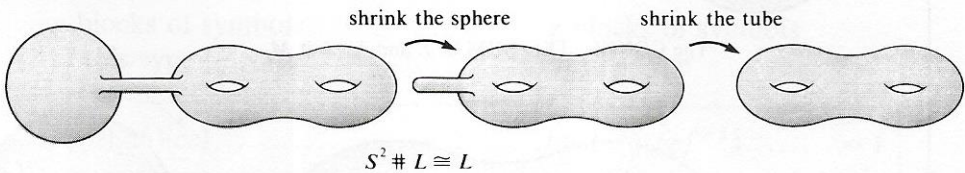
Alternative notation for connected sum



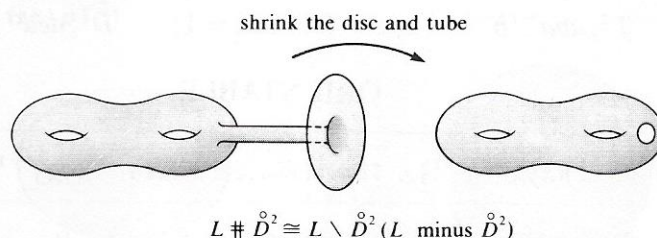
(∂ = boundary of each cut out disc)

FRAME 10

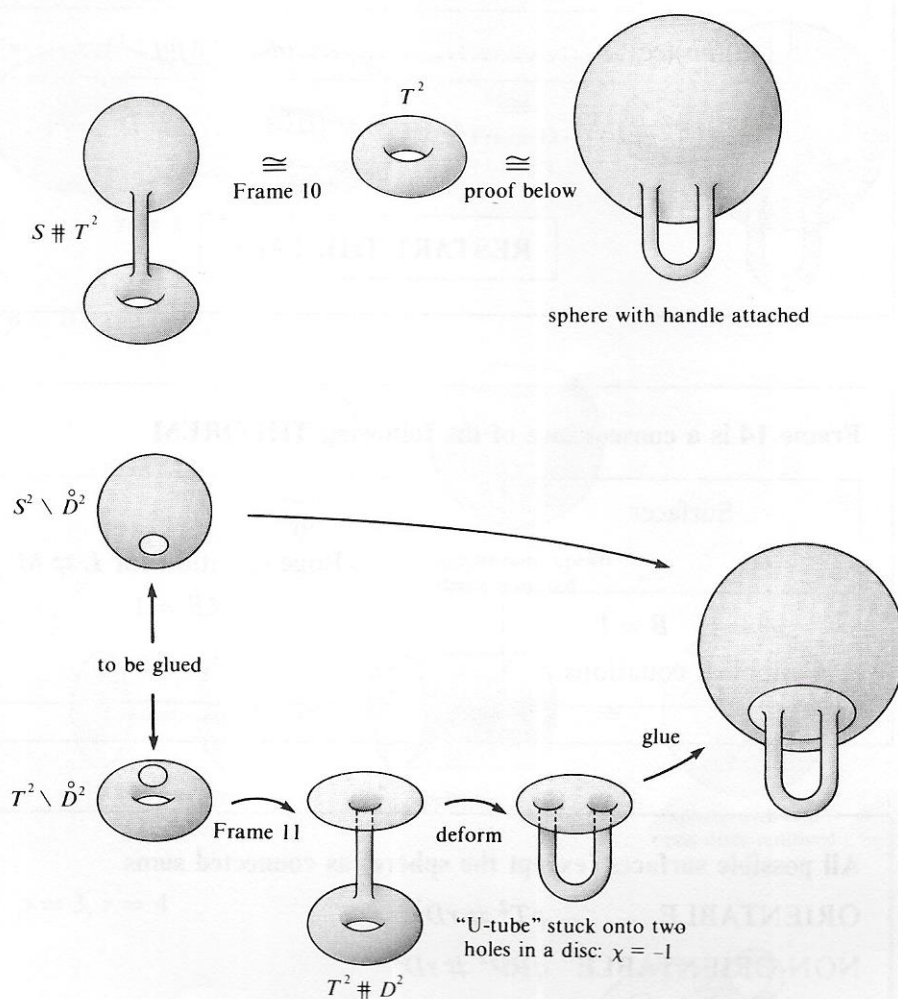
Connected sum with a sphere



FRAME 11

Connected sum with a disc

FRAME 12

For $L \# T^2$, attach a handle to L 

FRAME 13

The connected sums represented by two canonical equations

$$(aba^{-1}b^{-1})(ded^{-1})(fgf^{-1}) = 1$$

$$(aa)(bb)(ded^{-1}) = 1$$

$$T^2 \# D^2 \# D^2$$

$$2\mathbb{RP}^2 \# D^2$$

(or $T^2 \# 2D^2$, for short)

Before restarting the tape, read through Frame 14

FRAME 14

Every surface (except one) is a connected sum involving

$$T^2: aba^{-1}b^{-1} = 1; \quad \mathbb{RP}^2: aa = 1; \quad D^2: ded^{-1} = 1.$$

ORIENTABLE

$(aba^{-1}b^{-1})(xyx^{-1}y^{-1})\dots\dots\dots(ded^{-1})(fgf^{-1})\dots\dots = 1$

$$(T^2 \# T^2 \# \dots\dots\dots \# T^2) \# (D^2 \# D^2 \# \dots\dots \# D^2)$$

i.e. $pT^2 \# rD^2$

NON-ORIENTABLE

$(aa)(bb)(cc)\dots\dots\dots(ded^{-1})(fgf^{-1})\dots\dots = 1$

$$(\mathbb{RP}^2 \# \dots\dots\dots \# \mathbb{RP}^2) \# (D^2 \# \dots\dots \# D^2) = 1$$

i.e. $q\mathbb{RP}^2 \# rD^2$

RESTART THE TAPE

FRAME 15

Frame 14 is a consequence of the following THEOREM

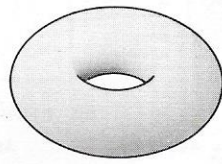
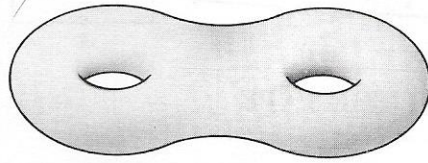
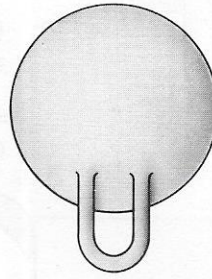
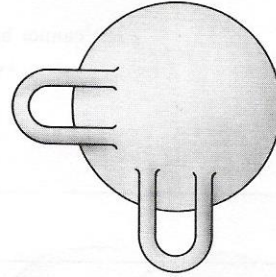
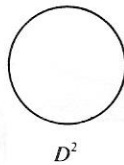
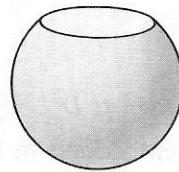
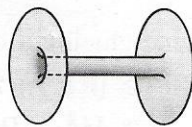
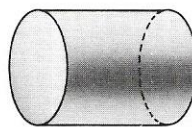
Surfaces $L$$M$ $A = 1, \quad B = 1$ Canonical equations	$\Rightarrow$	Edge equation for $L \# M$ $AB = 1$
---	---------------	---

FRAME 16

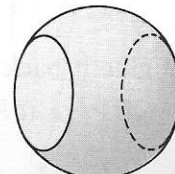
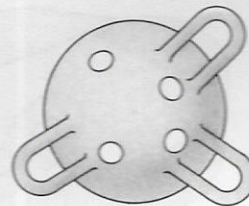
All possible surfaces (except the sphere) as connected sums

- ORIENTABLE $pT^2 \# rD^2$
- NON-ORIENTABLE $q\mathbb{RP}^2 \# rD^2$

FRAME 17

Orientable surfaces: $pT^2 \# rD^2$ (a) $r = 0$ (i.e. pT^2) T^2
 $\cong$
 Frame 12
 $(L = S^2)$
 $T^2 \# T^2 = 2T^2$
 $\cong$
 Frame 12
 $(L = S^2)$
(b) $p = 0$ (i.e. rD^2) D^2 $\cong$ sphere with open
disc removed $D^2 \# D^2 = 2D^2$ $\cong$ 

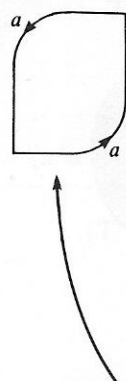
cylinder

 $\cong$ sphere with two
open discs removed(c) $p = 3, r = 4$  $3T^2 \# 4D^2$ $\cong$ 

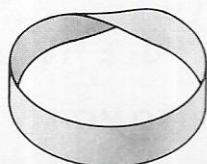
RESTART THE TAPE

FRAME 18

Non-orientable surfaces: $q\mathbb{RP}^2 \# rD^2$

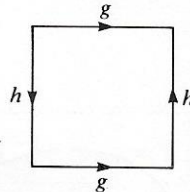


$$\begin{aligned} \mathbb{RP}^2 \# D^2 \\ \cong \mathbb{RP}^2 \setminus \overset{\circ}{D}^2 \\ \cong M^2 \\ \text{(pre-TV work)} \end{aligned}$$



$q\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$

$$\begin{aligned} \mathbb{RP}^2 \# \mathbb{RP}^2 \\ \cong (\mathbb{RP}^2 \setminus \overset{\circ}{D}^2) \cup_{\partial} (\mathbb{RP}^2 \setminus \overset{\circ}{D}^2) \\ \cong M^2 \cup_{\partial} M^2 \\ \cong K^2 \text{ (pre-TV work)} \end{aligned}$$



RESTART THE TAPE

FRAME 19

Visualizing $q\mathbb{RP}^2$ in $\mathbb{R}^3$

To visualize $q\mathbb{RP}^2$

Remove an open disc:

$$q\mathbb{RP}^2 \setminus \overset{\circ}{D}^2 \cong q\mathbb{RP}^2 \# D^2$$

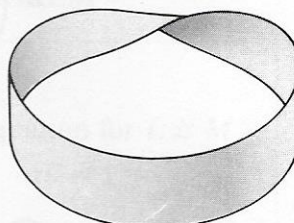
(This can always be embedded in $\mathbb{R}^3$, as you will see.)

and glue it back on:

$$q\mathbb{RP}^2 \cong (q\mathbb{RP}^2 \# D^2) \cup_{\partial} D^2.$$

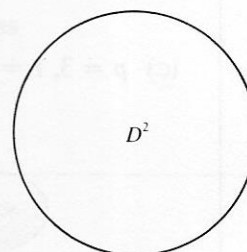
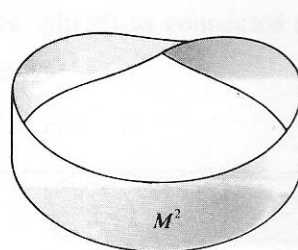
Example for $q = 1$

$$\mathbb{RP}^2 \# D^2 \cong M^2$$



Imagine glueing a disc back on:

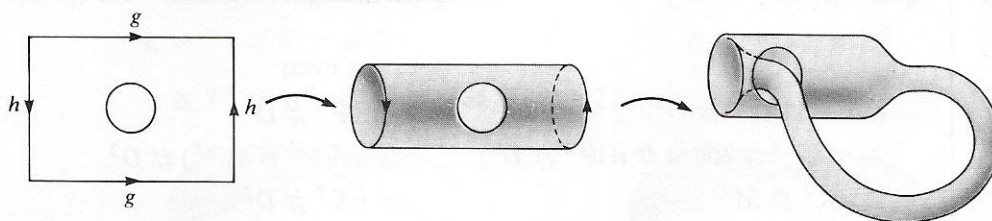
$$\begin{aligned} \mathbb{RP}^2 &\cong (\mathbb{RP}^2 \# D^2) \cup_{\partial} D^2 \\ &\cong M^2 \cup_{\partial} D^2 \end{aligned}$$



RESTART THE TAPE

FRAME 20

$2\mathbb{RP}^2 \# D^2$ can be embedded in $\mathbb{R}^3$

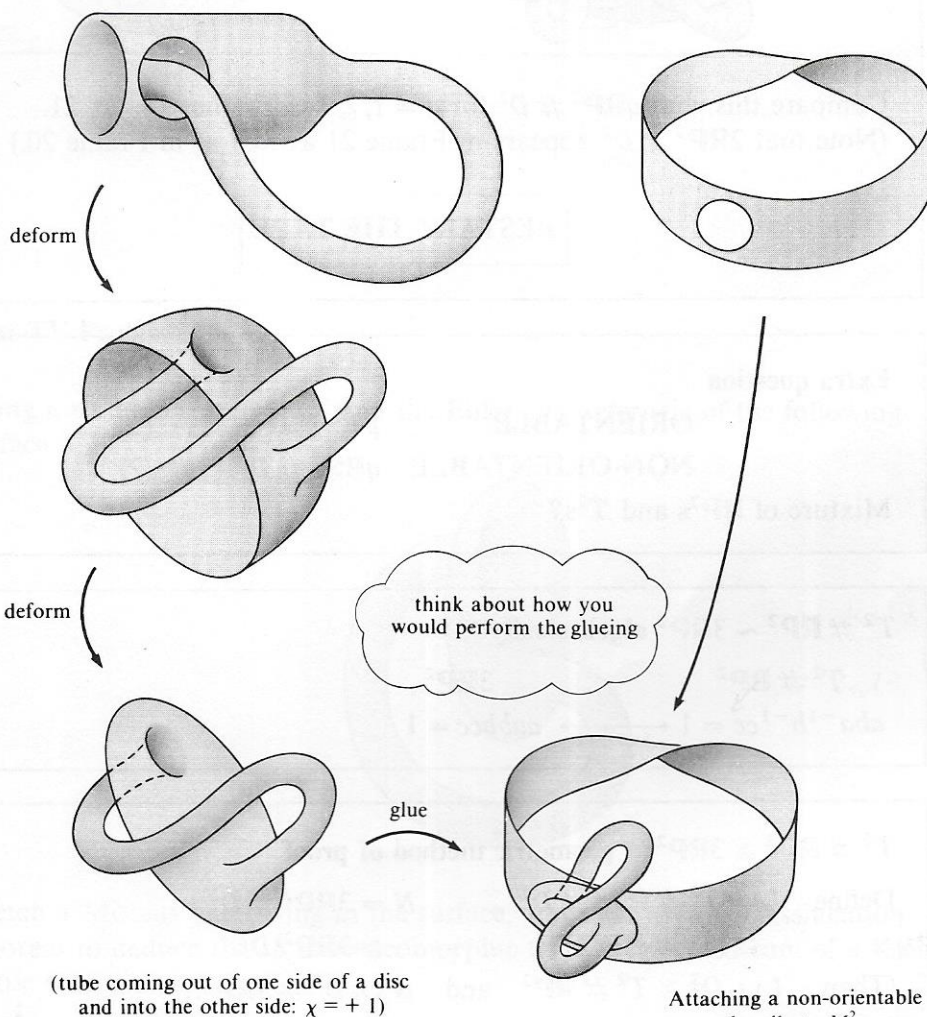


$$2\mathbb{RP}^2 \# D^2 \cong K^2 \# D^2 \cong K^2 \setminus \mathring{D}^2$$

FRAME 21

$3\mathbb{RP}^2 \# D^2$

$$\begin{aligned} &\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \# D^2 \\ &\cong (\mathbb{RP}^2 \# \mathbb{RP}^2) \# (\mathbb{RP}^2 \# D^2) \\ &\cong K^2 \# M^2 \\ &\cong K^2 \setminus \mathring{D}^2 \cup_{\partial} M^2 \setminus \mathring{D}^2 \end{aligned}$$



RESTART THE TAPE

FRAME 22

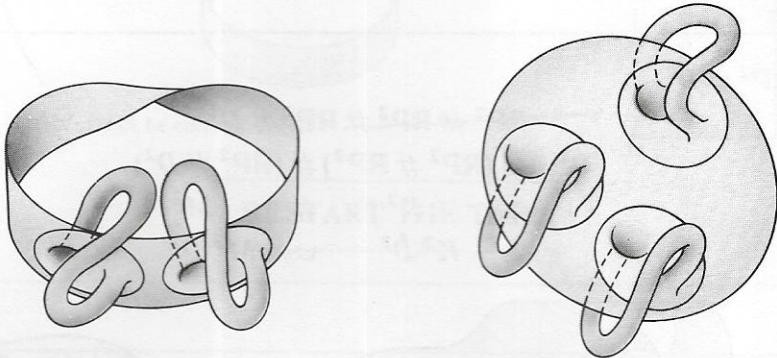
$q\mathbb{R}P^2 \# rD^2, \quad r = 1$

q odd
 $(2n + 1)\mathbb{R}P^2 \# D^2$
 $\cong n(\mathbb{R}P^2 \# \mathbb{R}P^2) \# (\mathbb{R}P^2 \# D^2)$
 $\cong nK^2 \# M^2$

q even
 $2n\mathbb{R}P^2 \# D^2$
 $\cong n(\mathbb{R}P^2 \# \mathbb{R}P^2) \# D^2$
 $\cong nK^2 \# D^2$

EXAMPLES

$2K^2 \# M^2 \cong 5\mathbb{R}P^2 \# D^2$ $3K^2 \# D^2 \cong 6\mathbb{R}P^2 \# D^2$



Compare this with $q\mathbb{R}P^2 \# D^2$ for $q = 1, 2, 3$ in Frames 18, 20, 21.
(Note that $2\mathbb{R}P^2 \# D^2$ appears in Frame 21 as well as in Frame 20.)

RESTART THE TAPE

FRAME 23

Extra question

ORIENTABLE	$pT^2 \# D^2$	(1)
NON-ORIENTABLE	$q\mathbb{R}P^2 \# D^2$	(2)

Mixture of $\mathbb{R}P^2$ s and T^2 s?

FRAME 24

$T^2 \# \mathbb{R}P^2 \sim 3\mathbb{R}P^2$ algebraically

$T^2 \# \mathbb{R}P^2$	$3\mathbb{R}P^2$
$aba^{-1}b^{-1}cc = 1$	$aabbcc = 1$

Section 4.2

FRAME 25

$T^2 \# \mathbb{R}P^2 \cong 3\mathbb{R}P^2$: geometric method of proof

Define $L = T^2 \# \mathbb{R}P^2 \# D^2$ $N = 3\mathbb{R}P^2 \# D^2$
 $\cong (T^2 \# \mathbb{R}P^2) \setminus D^2$ $\cong 3\mathbb{R}P^2 \setminus D^2$.

(Then $L \cup_{\partial} D^2 \cong T^2 \# \mathbb{R}P^2$ and $N \cup_{\partial} D^2 \cong 3\mathbb{R}P^2$.)

Prove $L \cong N$ (geometrically),
then it follows that $L \cup_{\partial} D^2 \cong N \cup_{\partial} D^2$
 $\parallel \qquad \qquad \parallel$
 $T^2 \# \mathbb{R}P^2 \qquad 3\mathbb{R}P^2$.

Before restarting the tape, try the proof in Frame 26

FRAME 26

 $T^2 \# \mathbb{RP}^2 \cong 3\mathbb{RP}^2$: geometric proof

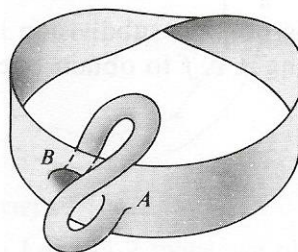
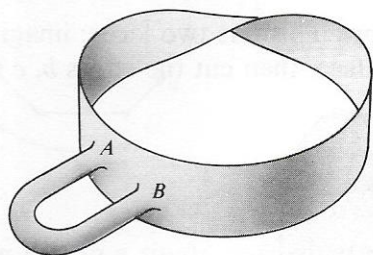
$$L = T^2 \# \mathbb{RP}^2 \# D^2$$

$$\cong T^2 \# M^2$$

cf. Frame 22

$$N = 3\mathbb{RP}^2 \# D^2$$

$$\cong K^2 \# M^2$$



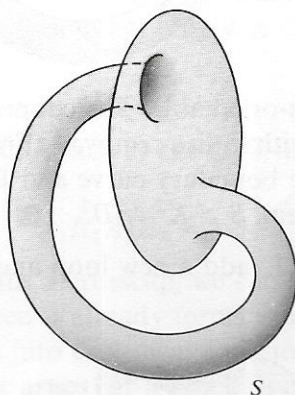
The result $T^2 \# \mathbb{RP}^2 \cong 3\mathbb{RP}^2$ follows, by Frame 25, if the first surface can be deformed homeomorphically into the second.

Try to visualize how this can be done.

RESTART THE TAPE

Post-TV Problems

- SAQ 11 Using a suitable subdivision, find the Euler characteristic of the following surface S .

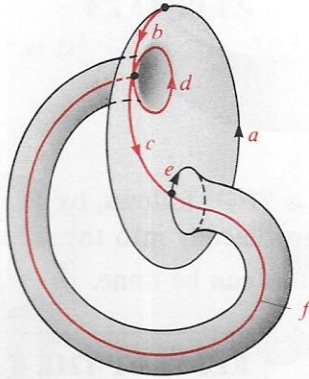


Sketch a Möbius band lying in the surface, and then use the classification theorem to deduce that S is homeomorphic to the connected sum of a Klein bottle with a disc.

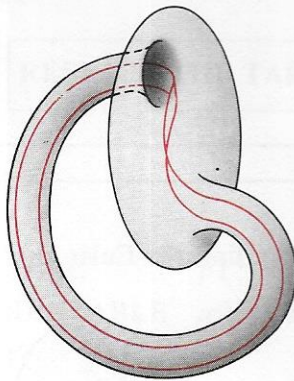
- SAQ 12 Surfaces L , M have edge equations $A = 1$, $B = 1$, respectively. By removing a disc from each and then glueing a cylinder into the “holes”, show that an edge equation for $L \# M$ is $AwBw^{-1} = 1$.
- SAQ 13 With the notation of SAQ 12, consider the special case when M is a disc and B is ded^{-1} . Show that the edge equation for $L \# M$ reduces to $AB = 1$.
- SAQ 14 Repeat SAQ 13 in the case when M is a projective plane, and B is cc .
- SAQ 15 Repeat SAQ 13 in the case when M is a torus, and B is $aba^{-1}b^{-1}$.

SAQ 16 Let L and M be surfaces which are represented by the edge equations $A = 1$ and $B = 1$, respectively, and suppose that both are in canonical form. (The blocks A and B have no symbol in common.) Let N be the surface represented by $AB = 1$. Calculate $\chi(N)$ in terms of $\chi(L)$ and $\chi(M)$.

SOLUTION TO SAQ 11 One possible subdivision is shown below. This has two faces; imagine cutting along d, e, f to obtain one rectangular face, then cut the edges b, c to convert

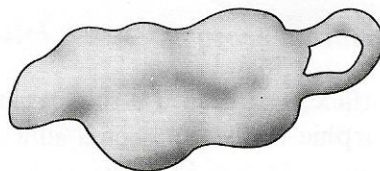


the remaining portion into a topological disc. There are six edges and three vertices, so the Euler characteristic is -1 . One example of a Möbius band is shown below.

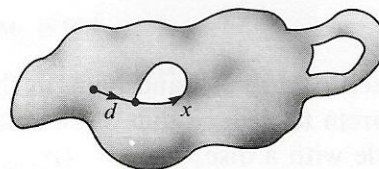


Thus the surface is non-orientable. The connected sum of a Klein bottle with a disc is a Klein bottle with a disc removed (Frame 11); and this is also non-orientable, and has one boundary curve and Euler characteristic -1 . Thus, by the classification theorem, $S \cong K^2 \# D^2$. ■

SOLUTION TO SAQ 12 To remove a disc from L , add a new loop and remove the interior of the region enclosed by this loop.

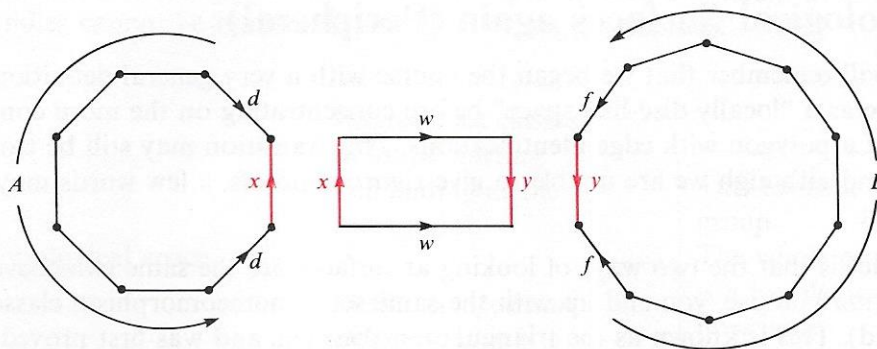


$$A = 1$$



$$Adxd^{-1} = 1$$

The resulting edge equation can be written as $Adxd^{-1} = 1$, where x is a boundary curve (x does not appear in A). The same argument works for M , giving a new boundary curve y , say. Now a cylinder is most easily represented as the edge equation $cwdw^{-1} = 1$, with boundary circles c and d . We can glue these ends to the edges x, y by labelling them accordingly. The resulting system of edge equations is



$$Adxd^{-1} = 1; \quad Bfyf^{-1} = 1; \quad xwyw^{-1} = 1.$$

Reducing this to a single edge equation, and using Lemma 2 twice, we obtain $AwBw^{-1} = 1$. ■

**SOLUTION
TO SAQ 13**

When $B = ded^{-1}$, the above form becomes

$$Awded^{-1}w^{-1} = 1 \xrightarrow{\text{Lemma 2}} Aded^{-1}ww^{-1} = 1$$

$$AB = 1. \quad \blacksquare$$

**SOLUTION
TO SAQ 14**

When $B = cc$ we obtain

$$Awccw^{-1} = 1 \xrightarrow{\text{Lemma 1}} Acw^{-1}cw^{-1} = 1$$

$$\xrightarrow{\text{Lemma 1}} Accww^{-1} = 1$$

$$AB = 1. \quad \blacksquare$$

**SOLUTION
TO SAQ 15**

When $B = aba^{-1}b^{-1}$ we find

$$Awaba^{-1}b^{-1}w^{-1} = 1 \xrightarrow{\text{Lemma 2}} Aaba^{-1}wb^{-1}w^{-1} = 1$$

$$\xrightarrow{\text{Lemma 2}} Aabwa^{-1}b^{-1}w^{-1} = 1$$

$$\xrightarrow{\text{Lemma 2}} Aawba^{-1}b^{-1}w^{-1} = 1$$

$$\xrightarrow{\text{Lemma 2}} Aaba^{-1}b^{-1}ww^{-1} = 1$$

$$AB = 1. \quad \blacksquare$$

**SOLUTION
TO SAQ 16**

Suppose that $A = 1$ and $B = 1$ are two edge equations in canonical form. Then we may represent them as $A_1A_2 = 1$, $B_1B_2 = 1$, where A_1, B_1 consist entirely of crosscaps or handles, and A_2, B_2 consist entirely of cuffs. Then N is represented by

$$A_1A_2B_1B_2 = 1.$$

We can move the block B_1 round each of the cuffs in A_2 (using Lemma 2) to obtain

$$A_1B_1A_2B_2 = 1.$$

Next, if either A_1 or B_1 contains a crosscap, we can convert the whole block A_1B_1 into crosscaps; if not, then it already forms a block of handles. Now, when inserting vertex symbols into canonical equations (cf. **B** page 79, lines 3–5), we found that one vertex arises for a block of crosscaps or handles, and an additional one for *each* cuff.

Thus the number of vertices of L is the number of cuffs in A_2 , v_a say, plus one. Similarly, the number of vertices of M is the number of cuffs in B_2 , v_b say, plus one. But the number of vertices in N is the number of cuffs in A_2B_2 (i.e. $v_a + v_b$), plus one. Thus we lose one vertex by moving from $A = 1$, $B = 1$, to $AB = 1$. We also lose a face, since there is now only one edge equation rather than two. But the number of edges (one for each symbol pair) is unaltered. Hence

$$\chi(N) = \chi(L) + \chi(M) - 2. \quad \blacksquare$$

The theme of connected sums is taken up again in the Exercise Booklet where there are many questions on both theory and practical calculation.

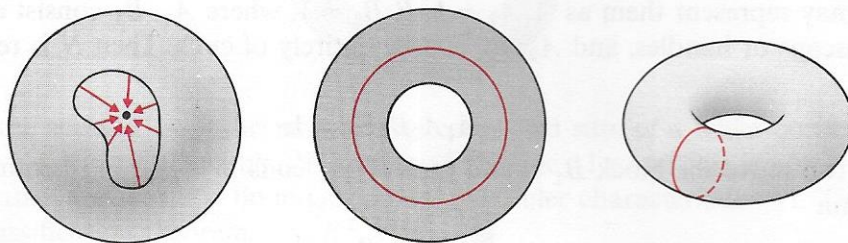
4.4 Topological Surfaces again (Peripheral)

You will remember that we began the course with a very general definition of a surface as a “locally disc-like space” before concentrating on the more concrete idea of a polygon with edge identifications. This transition may still be troubling you, and although we are unable to give rigorous proofs, a few words may be of interest.

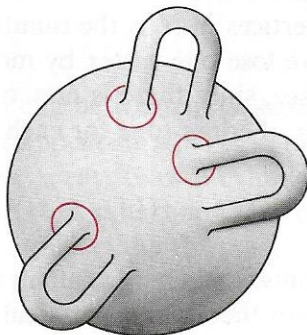
The fact is that the two ways of looking at surfaces are the same (whichever definition you use, you end up with the same set of homeomorphism classes in the end). This is known as the triangulation theorem, and was first proved by Rado in 1925. Roughly speaking, the triangulation theorem says that any locally disc-like space can be subdivided into triangles in the obvious “nice” way; that is, such that if any two triangles meet, then they do so in a single edge or a single vertex. It is not an easy theorem to prove, and the proof involves ideas which are similar to those needed to prove another “obvious” theorem in topology — the Jordan curve theorem. (This says, roughly speaking, that a simple loop in the plane divides the plane into two regions: the inside and the outside.)

In the classification theorem, we have proved that every triangulated surface is combinatorially equivalent to one of a list of standard surfaces. However, we have not shown that these standard surfaces are topologically all distinct. The simplest way to prove this is to show that the three properties (orientability, number of boundary curves, Euler characteristic) are all topological invariants. The methods used in this kind of proof form the basis of the subject known as *algebraic topology*. We shall give you a brief glimpse of the kind of ideas involved and then present a set of references for further reading if you are interested.

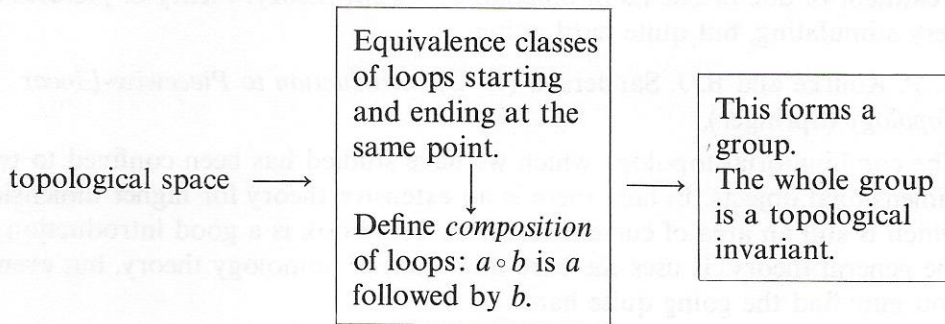
One starting point is to consider loops drawn on a surface. On a sphere, any loop can be continuously shrunk to a point without leaving the surface. This is not true of an annulus or torus.



The way forward is to consider equivalence classes of loops, two loops being considered equivalent if one can be continuously deformed into the other without leaving the surface. On a sphere, every loop is equivalent to the trivial loop consisting of a single point. On a torus or an annulus there are infinitely many loops which are inequivalent! (You will find further details in *M203 Unit 29*.) Moving to a sphere with several handles, it is easy to imagine loops which are not equivalent.



This can be developed to show that two spheres with a different number of handles cannot be homeomorphic. The strategy is depicted below.



This means that every space has associated with it a group — called the *fundamental group* — and homeomorphic spaces have the same fundamental group. So if two spaces have *different* fundamental groups, they *cannot* be homeomorphic.

This is a good start, but it does not deal with orientability and number of boundary curves. What we have to do is to construct some more groups from a topological space, called *homology groups*, which, like the fundamental groups, are topological invariants. These groups enable us to prove that the boundary of a surface, as defined in *Unit 1*, is a topological property (that is, a homeomorphism cannot map a boundary point of one surface into a non-boundary point of another surface), and to give a precise formulation of the key idea we need in defining orientability. Since homology groups can give a very simple proof of the Jordan curve theorem, and also prove that the Euler characteristic is a topological invariant, they are extremely useful, and the study of homology groups and the fundamental group forms the usual starting point of any advanced work in topology.

For Further Reading

Michael Henle (1979) *A Combinatorial Introduction to Topology* (Freeman).

A book which covers much the same ground as **B**, but laying more emphasis on point-set topology. It gives a proof of the Jordan curve theorem and an introduction to homology groups, but does not prove the triangulation theorem or the invariance theorems.

M. A. Armstrong (1978) *Basic Topology* (McGraw Hill).

A slightly more advanced book, nicely written with lots of examples. Narrower in scope than Henle, it classifies surfaces without boundary and proves that no two surfaces in the classification are homeomorphic, using the ideas we have hinted at above. It also has a section on knots, which you may have met in *M203*.

C. O. Christenson (1977) *Aspects of Topology* (Marcel Dekker & Voxman).

A rather more advanced book, harder to read. Not to be tackled unless you have got on well with *MTS* (which it summarizes nicely in the first hundred pages). Proves the triangulation theorem and the Jordan curve theorem, but does not really overlap with this course. If you have enjoyed *MTS*, quite a good book to dip into, with a very useful bibliography.

H. Seifert and W. Threlfall (1980) *A Textbook of Topology* (Academic Press).

The classic book that most treatments of classification of surfaces, homology theory, and so on, look back to. First published in 1937, it has never really been surpassed, in spite of many attempts. But, of course, the subject has progressed since.

D. Rolfsen (1976) *Knots and Links* (Publish or Perish, Inc.).

This contains a short proof of the Jordan curve theorem and is also a good treatment of one of our main omissions — knot theory. Plenty of pictures and very stimulating, but quite hard going.

C. P. Rourke and B. J. Sanderson (1972) *Introduction to Piecewise-Linear Topology* (Springer).

The combinatorial topology which we have studied has been confined to two-dimensional objects. In fact, there is an extensive theory for higher dimensions, which is still an area of current research. This book is a good introduction to the general theory. It uses the bare minimum of homology theory, but even so you may find the going quite hard.

Finally, for books that give a thorough treatment but which are really postgraduate reference books, we mention two for completeness. They are extremely difficult to read, but if you want a full treatment of topics like orientability there is not much alternative.

A. Dold (1972) *Lectures on Algebraic Topology* (Springer-Verlag).

E. H. Spanier (1966) *Algebraic Topology* (McGraw Hill).

